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Calcul stochastique dans les variétés et applications aux inégalités fonctionnelles

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Calcul stochastique dans les variétés et application aux inégalités fonctionnelles

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Résumé : Cette thèse explore les liens entre le calcul stochastique et l'analyse, dans un cadre géométrique riemannien. Nous nous attelons à étendre des résultats connus et des méthodes rodées, pour l'espace euclidien $\mathbb{R}^n$, en de nouveaux résultats et méthodes pour les variétés riemanniennes.

Les interactions considérés dans cette thèse seront de deux natures. D'une part, nous étudions l'interprétation stochastique des semi-groupes, de l'équation de la chaleur et ses applications aux inégalités fonctionnelles telles que Poincaré and FKG. Nous étudions les entrelacements entre diffusion et transport parallèle déformé, entre générateurs et entre semi-groupes. Le critère classique assurant ces relations est le critère de Bakry-Émery. Notre contribution principale est une généralisation de ce critère par la méthode de torsion (*twisting*). Nos donnons une condition générale pour obtenir des résultats d'entrelacement, d'inégalité fonctionnelle ou de trou spectral. Nous présentons comment utiliser ce résultat théorique sur des exemples explicites. Notre méthode illustre alors son efficacité en améliorant les résultats précédant sur les mesures de Cauchy généralisée.

D'autre part, nous étudions le problème de Brenier-Schrödinger, vu comme la relaxation du problème de minimisation associé aux équations de Navier-Stokes. Notre étude se place dans le cadre des variétés compactes à bords et nous traitons deux principales questions : les solutions du problèmes de Brenier-Schrödinger sont-elles solutions (et en quel sens?) des équations de Navier-Stokes et le problème de Brenier-Schrödinger admet-il une (unique?) solution? Ce travail généralise des résultats précédents dans le cadre euclidien ou le cadre du tore $\mathbb{T}^n$. Nos deux principales contributions sont l'étude du comportement des vitesses aux frontières du domaine et la méthode de quotient qui permet d'obtenir des espaces sur lequel le problème de Brenier-Schrödinger incompressible admet une unique solution.

Mots-clés : variété riemannienne, semi-groupe, diffusion, entrelacement, inégalité de Poincaré, problème de Brenier-Schrödinger

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Stochastic calculus on manifold and application to functional inequalities

Abstract: This thesis explores the links between stochastic calculus and analysis, in a Riemannian geometric framework. We are working on extending known results and tried and tested methods for the Euclidean space $\mathbb{R}^n$ into new results and methods for Riemannian manifolds.

We consider two kinds of interactions. On the one hand, we study the stochastic interpretation of semi-groups and its applications to functional inequalities such as Poincaré and FKG. We study intertwining relations between diffusion and deformed parallel transport, between generators and between semi-groups. The classical criterion ensuring these relations is the Bakry-Émery criterion. Our main contribution is a generalisation of this criterion by the twisting method. We give a general condition to obtain intertwining, functional inequality and spectral gap results. We present how to use this theoretical result on explicit examples. Our method illustrates its efficiency by improving previously known results on generalized Cauchy measures.

On the other hand, we study the Brenier-Schrödinger problem, seen as a relaxation of the minimization problem associated with Navier-Stokes equations. Our study takes place within the framework of compact manifolds with boundaries and we address two main questions. Are the solutions of the Brenier-Schrödinger problem solutions of the Navier-Stokes equations and in which sense? Does the Brenier-Schrödinger problem admit a (unique?) solution? This work generalises previously known results on the Euclidean and torus framework. Our two main contributions are the study of the behaviour of velocities at the boundaries of the domain and the quotient method which allows to obtain spaces on which the incompressible Brenier-Schrödinger problem admits a unique solution.

Keywords: Riemannian manifold, semi-group, diffusion, intertwining, Poincaré inequality, Brenier-Schrödinger problem

Contents

Part I

Introduction

"Arma dedi vobis : dederat Vulcanus Achilli;
Vincite numeribus, vicit ut ille, datis."
Ovide, *Artis Amatoriae*

Introduction en français

Nous présentons les deux axes de cette thèse : les entrelacement appliqués aux inégalités fonctionnelles et le problème de Brenier-Schrödinger. En partant de l'état de l'art, nous expliquons notre approche et nos résultats.

Introduction

Cette thèse explore les liens entre le calcul stochastique et l'analyse. Les interactions considérés dans cette thèse seront de deux natures. D'une part, il y a l'interprétation stochastique des semi-groupes, de l'équation de la chaleur et ses applications aux inégalités fonctionnelles telles que Poincaré and FKG. Ses questions seront traitées en Partie II font l'objet de l'article [47]. D'autre part, il y a la relaxation de problèmes variationnels. Ces problèmes sont liés à la dualité entre mécanique classique (seconde loi de Newton) et mécanique analytique (principe de moindre action). Ces questions seront traitées en Partis III et dans l'article [42].

Pour chacun des deux sujets, notre travail se place dans le cadre de géométrie riemannienne. Nous nous attelons à étendre des résultats connus et des méthodes rodées, pour l'espace euclidien $\mathbb{R}^n$, en de nouveaux résultats et méthodes pour les variétés riemanniennes. Si la plupart de nos interrogations se prêtent bien au passage riemannien, certaines notions, tel que la croissance au Chapitre 5, sont fortement dépendantes de la structure vectorielle et ne sont pas aisées à traduire.

Le but de ce chapitre est de présenter les deux problèmes étudiés dans cette thèse ainsi que notre contribution.

Entrelacement et inégalité fonctionnelle

S.C.E.P. : Semi-groupe, Chaleur, Exponentielle, Pollen

Partant d'un opérateur différentiel de second ordre, noté L, deux chemins s'offrent à nous : le probabiliste et l'analytique. Le premier nous conduit vers un processus stochastique, la diffusion de générateur L. Le second, à travers l'équation

$\partial_t u = Lu$, mène au semi-groupe $\mathbf{P}_t = e^{tL}$. Une représentation stochastique du semi-groupe, faisant intervenir la diffusion, boucle le chemin. Nous avons décrit ici des liens entre trois types d'objet : la diffusion, le générateur et le semi-groupe. L'illustration classique des ces relations est le triplet mouvement brownien, laplacien and semi-groupe de la chaleur. Ces relations donnent lieu à des transferts de propriétés entre les différents niveaux, en particulier, entre la convergence de la loi du processus stochastique vers sa mesure réversible μ, les propriétés spectrale du générateur et la convergence à l'équilibre du semi-groupe.

Dans cette thèse, nous nous intéressons à l'opérateur différentiel

$$L = \Delta - \langle \nabla V, \nabla \cdot \rangle,$$

où V est un potentiel lisse, défini sur une variété riemannienne M. Nous recherchons des résultats sur le spectre de L, ou bien des inégalités fonctionnelles sur Cov_μ, en utilisant les liens entre ces trois niveaux. Notre approche se base sur les entrelacements.

Entrelacement

L'idée de base des entrelacements est de pouvoir réécrire la différentielle du semi-groupe agissant sur les fonctions, $d\mathbf{P}f$ comme l'action d'un semi-groupe $\mathbf{Q}$ agissant sur la forme différentielle df :

$$d\mathbf{P}f = \mathbf{Q}df.$$

Les semi-groupes $\mathbf{P}$ est $\mathbf{Q}$ sont alors entrelacés. L'enjeu est alors d'étudier l'action de la différentielle sur les deux autres niveaux. Ces relations sont au cœur des travaux de thèse de Xue-Mei Li [58] et de ses travaux ultérieurs (Cf [59] ou [60]). L'étude des ses relations appliquée aux inégalités fonctionnelles a été abordée dans [25] pour les processus de vie ou de mort et dans [18] pour le cas unidimensionnel. Les diffusions réversibles et ergodiques dans $\mathbb{R}^n$ font l'objet de l'article [2]. Notons que le concept d'entrelacement étaient déjà sous-jacent dans l'étude du Γ_2 de Bakry, avec des formules de sous-commutation entre $|\nabla \mathbf{P}f|$ et $\mathbf{P}|\nabla f|$ (Cf [8]).

Sur les deux premiers niveaux, les relations d'entrelacement sont réalisées sans plus d'hypothèses. Le générateur L est entrelacé à un laplacien à poids agissant sur le 1-formes, noté L^W. Une étude approfondie de ce générateur peut être trouvé dans les travaux de Hellfer, avec une application à la décroissance de la corrélation dans les systèmes de spin ([45]). Au niveau des processus stochastique, L^W est le générateur, sur les 1-formes, du transport parallèle déformé. Dans [3], ce processus est interprété comme la dérivée spatiale d'un flot convenable de la L-diffusion. Ces entrelacements aux niveaux des processus et des générateurs suggèrent l'entrelacement au niveau des semi-groupes ainsi qu'une représentation stochastique de cet entrelacement. Cependant, cela n'est vraiment pas automatique. Si le semi-groupe $\mathbf{Q}$ peut toujours être défini au sens L^2, sa représentation stochastique ou la relation d'entrelacement ne sont pas nécessairement valides. La situation se résume ainsi.

$$\begin{tikzcd}
X_t \arrow[rrr, "d"] \arrow[dd, "\mathbb{E}"'] & & & W_t \arrow[dd, dotted, "?"] \\
& L \arrow[ul] \arrow[dl, "e^{t\cdot}"'] \arrow[r, "d"] & L^W \arrow[ur] \arrow[dr, "e^{t\cdot}"] & \\
\mathbf{P}_t \arrow[rrr, dotted, "?"] & & & \mathbf{Q}_t
\end{tikzcd}$$

Le critère de Bakry-Émery est une hypothèse classique et utile garantissant l'existence du $\mathcal{C}^0$-semi-groupe $\mathbf{Q}$, la relation d'entrelacement et des inégalité fonctionnelles classiques (Poincaré, Log-Sobolev). Il s'agit d'une condition de minoration de la partie potentiel de L^W. Dans $\mathbb{R}$. ce potentiel est la hessienne de V. Dans une variété riemannienne, la géométrie agit via le courbure de Ricci. Nous nous intéressons au cas où ce critère n'est pas vérifié. Est-il toujours possible d'obtenir des inégalités fonctionnelles? Une réponse est proposée à travers les entrelacements tordus, exposés dans [2]. Le principe est le suivant : si l'on ne peut pas montrer que la différentielle $d\mathbf{P}$ est entrelacé alors remplaçons d par une différentielle tordue $(B^*)^{-1}d$, où B est une section de $\mathrm{GL}(TM)$. Manifestement, le semi-groupe entrelacé en sera plus $\mathbf{Q}$ mais un nouveau dont on espère tirer profit. Le but est de trouver une torsion B idoine, permettant au nouveau semi-groupe $\mathbf{Q}^B$ de vérifier un critère de Bakry-Émery généralisé. Sous des hypothèses de symétrie et des contraintes sur les valeurs propres, une inégalité de Brascamp-Lieb généralisée est démontrée dans [2], avec des exemples concrets.

Contributions de cette thèse

Dans cette thèse, nous présentons le transport parallèle déformé comme un outil pour obtenir des entrelacements. Dans le Chapitre 3, il permet de retrouver des démonstration entièrement probabilistes de résultats classiques sous le critère de Bakry-Émery, comme l'inégalité de Brascamp-Lieb asymétrique du Théorème 3.4.4. Notons $\mathcal{M}$ la partie potentiel de L^W et ρ un minorant de $\mathcal{M}$.

Théorème 1 (Théorème 3.4.4). *Si $\rho > 0$, alors pour toutes $f, g \in \mathcal{C}_c^\infty(M)$, on a :*

$$\mathrm{Cov}_\mu(f,g)| \leq \frac{1}{\rho}\|dg\|_\infty \int_M |df|\, d\mu.$$

Poursuivant le travail de [2], dans le Chapitre 4, nous généralisons la méthode de torsion (*twisting*) aux variétés riemannienne et à une plus vaste classe de torsion. Nous rattachons cette méthode à l'étude du transport parallèle déformé tordu. Cela donne une nouvelle interprétation aux opérateurs en jeu. Notre principale résultat est un critère de Bakry-Émery généralisé et l'inégalité de Brascamp-Lieb du théorème 4.5.4. Notons $\mathbb{M}_B$ le potentiel tordu et $\mathcal{B}$ l'opérateur caractéristique de la symétrie.

Théorème 2 (Théorème 4.5.3)**.** *Si $\mathcal{B} = 0$ et M_B est défini positif, alors pour toute $f \in \mathcal{C}_0^\infty(M)$, on a :*

$$\mathrm{Var}_\mu(f) \leq \int_M \langle df, \left(B^* M_B (B^*)^{-1}\right)^{-1} df\rangle \, d\mu.$$

Les hypothèses de ce théorème peuvent être affinées (Théorème 4.6.3) pour obtenir un énoncé stable sous perturbations. Ces résultats sont illustrés par plusieurs exemples de modèle variété-mesure pour lesquels le critère de Bakry-Émery n'est pas vérifié, ou pas optimal, de différente manière. L'exemple le plus notable est celui des mesures de Cauchy généralisées $\mathbb{R}^2$. Nous améliorons les bornes précédemment connues du trou spectral (Corollaire 4.7.1).

Le Chapitre **??** s'intéresse à un autre type d'inégalité fonctionnelle : l'inégalité FKG. Dans l'esprit de [39], nous donnons ue démonstration stochastique du lien avec la préservation de la monotonicité par le semi-groupe de la chaleur. Cette notion peut être étendue aux groupes de Lie et nous obtenons un critère similaire sur ces espaces (Théorème 5.5.3) .

Problème de Brenier-Schrödinger

Une approche variationnelle de Navier-Stokes

Un même phénomène physique peut être décrit de manière for différentes. A contempler le cours de la blonde Garonne, d'aucuns verront un champ de vitesse, solution d'une équation newtonienne, quand d'autres observeront un flot paresseux, minimisant ses efforts à travers le port de la Lune. Les premiers ont une visons eulérienne de l'écoulement : ils décrivent l'évolution par son champ de vitesse et par les équations qu'il vérifie. Ces équations ont été introduite par Euler au XVIII^e siècle pour des fluides parfaits. Ce n'est que près d'un siècle plus tard que sera introduit la notion de viscosité par Navier et Stokes. Ces questions sont notoirement difficiles et la résolution des équations de Navier-Stokes est un problème du prix du millénaire. Notre approche de ces équations n'a ni pour but ni la prétention de les résoudre.

Les seconds ont une vision lagrangienne de l'écoulement : ils décrivent l'évolution par les trajectoires de chaque particules dont le mouvement respect un principe de moindre action. Cette approche physique a été appliquée aux équation d'Euler par Arnold dans [7] et aux équations de Navier-Stokes par Yasue dans [77]. Dans ce dernier cas, l'action minimisée est une version stochastique de l'énergie cinétique, définie avec la vitesse de Nelson. Sous cette formulation, ces problèmes restent ardue et il existe plus de résultats de non-existence au problème d'Arnold ([73] ou [74]) que d'existence.

Adirant le cours de la rivière, il y a un troisième groupe de personnes. Ils n'y voient pas un champ de vitesse, ils n'y voient pas une trajectoire. Ils observe une distribution de trajectoires. Brenier introduisit en premier cette relaxation du problème d'Arnold dans [21]. Une solution P du problème de Brenier est une mesure sur les trajectoires minimisant l'énergie cinétique moyenne

$$\mathbb{E}_P\left[\int_0^1 \dot{X}_t \, dt\right],$$

tout en respectant une contrainte de préservation de volume sur ses marginale et une prescription du couplage P_{01}. Il obtient une bonne correspondance entre les solutions de son problème et les solutions des équations d'Euler. Dans cette thèse, nous nous intéressons à une généralisation du problème de Brenier pour prendre en compte la viscosité.

Problème de Brenier-Schrödinger

Les équation de Navier-Stokes sont le système différentiel suivant :

$$\begin{cases} \partial_t v + \nabla_v v - a\square v + \nabla p = 0, & (t,x) \in [0,1] \times M \\ \operatorname{div}(v) = 0, & (t,x) \in [0,1] \times M \\ \langle v, \nu \rangle = 0, & (t,x) \in [0,1] \times \partial M \\ v(0,\cdot) = v_0, & x \in M \end{cases}$$

où M est une variété riemannienne compacte à bord„ ν un champ de vecteur dont la restriction à ∂M est le vecteur normal entrant et v_0 une condition initiale. Les inconnues de ce système sont la pression p et le champ de vitesse v. Ces équations peuvent être vues comme une perturbation des équations d'Euler par un terme de viscosité $-a\square v$.

La première généralisation des idées de Brenier aux équations de Navier-Stokes vient de [1]. Le problème est alors présenté comme une minimisation d'une énergie cinétique moyenne, sur les loi de mouvement brownien avec dérive, avec des contraintes de marginale. Cette énergie cinétique stochastique est alors définie avec la dérive en lieu et place de la vitesse classique. Depuis cet article, de nombreux auteurs se sont penchés sur le problème, lui donnant sa formulation actuelle et son nom. Parmi eux, je pense à [4], [15], [16], [17], les thèse de Baradat [14] et Nenna [66] ainsi que l'article [5] qui inspire notre travail.

L'énergie cinétique moyenne de [1] peut être exprimée comme une entropie relative au mouvement brownien. Le problème devient donc une minimisation de l'entropie sur les mesures de probabilité sur l'espace des chemins Ω, sous contraintes de marginale. Il se formule ainsi :

$$H(Q|R) \to \min,\ Q \in \mathcal{P}(\Omega),\ [Q_t = \mu_t,\ \forall t \in \mathcal{T}],\ Q_{01} = \pi.$$

où $\pi \in \mathcal{P}(M^2)$ est la mesure sur les extrémités et $\mu_t \in \mathcal{P}(M)$ les contraintes de marginale. Ainsi exprimé, les prioblème est un mix entre le problème de Brenier et le problème de Schrödinger.

En tant que problème en soit, il y a trois questions principales :

a) Admet-il une (unique) solution?

b) Quelles sont les caractéristique des solutions?

c) Peut-on faire de simulations numériques?

L'interprétation du problème comme une minimisation d'entropie sous contrainte linéaire apporte des réponses à ses trois questions. En particulier, dans [5], il est démontré que le problème incompressible sur le tore $\mathbb{T}^n$ admet une unique solution si et seulement si une condition d'entropie finie est satisfaite par la mesure π.

En tant qu'approche variationnelle des équations de Navier-Stokes, il y a deux principales questions :

d) Une solution de Navier-Stokes est-elle solution du problème de Brenier-Schrödinger (et en quel sens)?

e) Une solution de Brenier-Schrödinger est-elle solution de Navier-Stokes (et en quel sens)?

Le cas de la question d) est traité dans [4] pour le tore $\mathbb{T}^n$. La question e) est traité dans [5] pour le tore et l'espace euclidien $\mathbb{R}^n$ avec la notion de solution régulière.

Contributions de cette thèse

Le but de notre travail est de généralisé l'étude du problème de Brenier-Schrödinger aux variétés compactes à bord. L'entropie relative est alors définie en référence à la mesure du mouvement brownien réfléchi. Nous nous intéressons aux questions a), b) et e).

La question de l'existence est le sujet du Chapitre 9. Nous généralisons l'argument pour le tore développé dans [5], aux espaces symétriques. Nous obtenons le même critère d'entropie sur π.

Corollaire 3 (Corollaire 9.2.2)**.** *Le problème de Brenier-Schrödinger incompressible sur M admet une unique solution si et seulement si $H(\pi|R_{01}) < \infty$.*

Nous développons ensuite une stratégie qui permet de passer des espaces symétriques à des variétés à bord via des quotients. Cela apporte de nombreux exemples tels que le triangle équilatéral, les parallélépipède de dimension n ou le disque. Enfin, nous étudions un problème plus exotique, sur $\mathbb{R}^n$, pour lequel les arguments de compacités ne sont plus utilisables.

La question de la caractérisation des solutions est l'objet du Chapitre 7. Nous montrons que les solutions sont des semi-martingale et nous reprouvons le lien entre le problème de minimisation d'entropie et le problème de minimisation de l'énergie cinétique.

Le Chapitre 8 traite de la question e). Notre principale apport est le comportement des solution sur la frontière du domaine. Nous montrons que la vitesse stochastique rétrograde vérifie la partie newtonienne des équations de Navier-Stokes ainsi que la condition d'imperméabilité.

Théorème 4 (Théorème 8.4.2)**.** *Pour P_0 presque tout $y \in M$, la vitesse stochastique rétrograde $\overleftarrow{v}^y$ vérifie :*

$$\begin{cases} \left(\partial_t + \nabla_{\overleftarrow{v}^y}\right) \overleftarrow{v}^y = \dfrac{a}{2}\square \overleftarrow{v}^y - \mathbb{1}_{\mathcal{T}}(t)\nabla(ap), & 0 \leq t < 1, t \notin \mathcal{S}, z \in M, \\ \overleftarrow{v}^y_t - \overleftarrow{v}^y_{t^-} = \theta_t(.), & t \in S, z \in M, \\ \langle \overleftarrow{v}^y, \nu(z)\rangle = 0, & z \in \partial M, \\ \overleftarrow{v}^y_0 = -\nabla\eta(., y), & z \in M. \end{cases}$$

De plus il existe un potentiel scalaire φ^y tel que

$$\overleftarrow{v}^P_t (X) = -a\nabla\varphi_t^{X_1}(X_t),\ P\text{-}p.s.$$

Nous montrons aussi une équation de continuité pour la vitesse de courant (Théorème 8.5.1).

Chapter 1

Synopsis and latest developments

We present the two main topics of this thesis : intertwining applied to functional inequalities and Brenier-Schrödinger problem. We recall the state of art and present our approach and results.

Contents

1.1 Introduction

This thesis examines interactions between stochastic calculus and analysis. The interactions considered in this thesis can be classified in two kinds. On the first side, we have the stochastic interpretation of semi-groups, heat equation and its applications to functional inequalities as Poincaré and FKG. These questions are the subject of Part II and of the article [47]. On the other side, we have relaxations of variational problem. Those problems are linked to the duality between classical mechanics (Newton's second law) and analytical mechanics (principle of least action). Those questions are the subject of Part III and of the article [42].

A shared axis of our work on these two topics is the manifold background. We work on extension of known results or methods, for the Euclidean space $\mathbb{R}^n$, to results and methods on Riemannian manifolds. If the majority of our questions lends themselves to a manifold translation, some notions, as increasing function in Chapter 5, rely on the vectorial structure and are not so easy to transfer.

The goal of this chapter is to present the two problems treated in our work and our main contributions.

1.2 Intertwining and functional inequalities

1.2.1 The Heat, the Seed and the Exponential

From a second order differential operator L, there are two paths : the probability one and the analysis one. The first one leads to a stochastic process, a diffusion with generator L. The second one leads, through the equation $\partial_t u = Lu$, to a semi-group $\mathbf{P}_t = e^{tL}$. A stochastic representation of the semi-group depending on the diffusion rounds everything off. This procedure links three levels of objects : the diffusion, the generator and the semi-group. A classical example of such relation is given by the Brownian motion, the Laplacian and the heat semi-group. There are transfers of properties between these levels, in particular between the convergence of the diffusion measure to its reversible measure μ, the spectral properties of the generator and the convergence to equilibrium of the semi-group.

In this thesis, we work on the differential operator

$$L = \Delta - \langle \nabla V, \nabla \cdot \rangle,$$

where V is a smooth potential, defined on a Riemannian manifold. We try to obtain some spectral results for L, or functional inequality on Cov_μ, using the links between the three levels. Our approach is based on the intertwining method.

1.2.2 Intertwining

The principle of intertwining is to study the action of differentiation on the three levels and especially at the semi-group level. The goal is to rewrite the derivative of a the semi-group acting on function $d\mathbf{P}_t f$ as a semi-group acting on differential forms $\mathbf{Q}_t df$. These relations form a important part of Xue-Mei Li PhD thesis [58] and work (see [59] or [60]). They have been applied to functional inequalities firstly in the discrete case for birth-death processes in [25] and in the one dimensional case in [18]. The case of reversible and ergodic diffusions in $\mathbb{R}^n$ is treated in [2]. Remark that the material of intertwining was already underlying in the Γ_2 theory with sub-commutation formulae between $|\nabla \mathbf{P}_t f|$ and $\mathbf{P}_t |\nabla f|$ as in [8].

At the two first levels, intertwining relation occurs without further assumptions. The generator L is intertwined to a weighted Laplacian acting on 1-forms, denoted L^W, unitary equivalent to the Witten Laplacian. A large study of this operator can be found in the work of Helffer, with applications to correlation decay in spin systems (see [45]). At the level of stochastic processes, L^W is the generator on 1-forms of the deformed parallel translation. In [3], this process is proved to be the spacial derivative of an appropriate flow of the L-diffusion. These intertwining relations at these two levels suggest an intertwining relation at the level of the semi-group and a stochastic representation of the intertwined semi-group. However, it is not automatic. The semi-group $\mathbf{Q}$ can be defined as a L^2 semi-group but its stochastic representation or the intertwining relation are not necessarily true. The situation can be resume as follow.

$$\begin{tikzcd}
X_t \arrow[rrr, "d"] \arrow[dd, "\mathbb{E}"'] & & & W_t \arrow[dd, dotted, "?"] \\
& L \arrow[ul] \arrow[dl, "e^{t\cdot}"'] \arrow[r, "d"] & L^W \arrow[ur] \arrow[dr, "e^{t\cdot}"] & \\
\mathbf{P}_t \arrow[rrr, dotted, "?"] & & & \mathbf{Q}_t
\end{tikzcd}$$

The Bakry-Émery criterion is a classical and handy assumption which guarantees the existence of the $\mathcal{C}^0$ semi-group $\mathbf{Q}$, intertwining relation and classical inequalities (Poincaré, Log-Sobolev). It is a lower-boundedness condition on the potential part of L^W. In the Euclidean case, this potential is the Hessian of V. In a manifold, the geometry plays a part through Ricci curvature. The interesting situation comes when Brakry-Émery criterion is not satisfied. Is it still possible to obtain spectral gap properties? An answer is given by the twisting approach developed in [2]. The idea is the following : if we cannot reach the intertwining relation with the differential d, let us use a twisted differential $(B^*)^{-1}d$, where B is a section of $\mathrm{GL}(TM)$. Obviously, the intertwined semi-group will no longer be $\mathbf{Q}$ but a new one. The goal is to find twist B such that the new semi-group has Bakry-Émery like properties. Under symmetry and eigenvalue conditions, a generalized Brascamp-Lieb inequality is proved in [2]. This approach is strongly linked to Lyapunov functions from [24], especially for homothetic twists.

1.2.3 Contributions of this thesis

In this thesis, we present the deformed parallel translation as a tool for intertwining. In Chapter 3, it allows us to develop only stochastic proofs of some well-known results, under Bakry-Émery criterion, as the asymmetric Brascamp-Lieb inequality of Theorem 3.4.4. We denote by $\mathcal{M}$ the potential part of L^W, $\rho(x)$ the smallest eigenvalue of $\mathcal{M}(x)$ and ρ its infimum on M.

Theorem 1.2.1 (Theorem 3.4.4)**.** *Assume that $\rho > 0$, then for all $f, g \in \mathcal{C}_c^\infty(M)$, we have :*

$$|\mathrm{Cov}_\mu(f,g)| \leq \frac{1}{\rho}\|dg\|_\infty \int_M |df|\, d\mu.$$

Following the work of [2], in Chapter 4, we generalise the twisting method, to Riemannian manifolds and to a wider class of twistings. Our approach starts from a twisted deformed parallel translation which shades a new light on the operators and potentials. The main result of this chapter is the generalised Brascamp-Lieb inequality of Theorem 4.5.4. We denote by M_B the potential of the intertwined generator and by $\mathcal{B}$ the operator characterising symmetry and positiveness :

$$\mathcal{B} = \left((\nabla B^*)(B^*)^{-1}\right)^t - (\nabla B^*)(B^*)^{-1}.$$

Theorem 1.2.2 (Theorem 4.5.3)**.** *Assume that $\mathcal{B} = 0$ and that M_B is positive definite, then for every $f \in \mathcal{C}_0^\infty(M)$, we have :*

$$\mathrm{Var}_\mu(f) \leq \int_M \langle df, \left(B^* M_B (B^*)^{-1}\right)^{-1} df\rangle\, d\mu.$$

The conditions of this theorem can be relaxed (Theorem 4.6.3). It shows the stability of the twisting method under perturbations. These results are illustrated by examples. We present cases where the Bakry-Émery criterion is not satisfied in different ways and we propose appropriate twistings. The most remarkable example is our Riemannian approach of Cauchy measure in $\mathbb{R}^2$. We obtain in Corollary 4.7.1 some improvement of the lower bound for the spectral gap. Our examples also show that even if Bakry-Émery criterion is satisfied, twistings can be used to obtain better bounds.

In the spirit of [13], we use intertwinings and deformed parallel translation to study the FKG inequality. We show its link to the preservation of positive vectors. This notion can be defined in Lie groups. We obtain a generalisation of this property for Brownian motion in Proposition 5.3.3 and for more general diffusions in Theorem 5.5.3.

1.3 Brenier-Schrödinger problem

1.3.1 A variational approach of Navier-Stokes

A same phenomenon can be describe from different points of view. Looking at our blonde Garonne, some will see a velocity field satisfying some differential system, while others will grasp a flow minimising its efforts in its way through the *port de la Lune*. The first ones have the Eulerian point of view. Its is a description of the flow by its velocity and by a differential system. Euler proposed, in 1757, his equations for incompressible inviscid fluids. It was not until the 19th century that the concept of viscosity was introduced into this model, by Navier in 1823 and Stokes in 1845. These problems are notoriously difficult and the resolution of Navier-Stokes equation is a Millennium Prize Problem. This is not the pretension nor the goal of our approach to solve it.

The second group has a Lagrangian point of view. They describe the flow by the trajectories of its particles, moving according to the least action principle. This general physical principle, firstly inspired by philosophical and theological conceptions, was mathematically formulated in 1756 and proved to be equivalent to Newton principles in 1788, by Lagrange. This approach have been applied to Euler equation by Arnold in [7]. In his track, Yasue give a variational formulation to Navier-Stokes equation in [77]. In his formulation, the action derived from a stochastic kinetic energy, defined with the Nelson velocity instead of the usual velocity. These problems, even with these formulations, stay arduous. There are few results of existence for Arnold problem ([31]) and more results of non-existence ([73] or [74]).

Staring at the river, there is a third group of persons. They do not see a velocity field. They do not see a trajectory. They see a distribution of trajectories. Brenier introduced this relaxation of Arnold problem in [21]. A solution P of his problem is a path measure minimizing the mean kinetic energy $\mathbb{E}_P[\int_0^1 |\dot{X}_t|^2\, dt]$, with a volume preservation constraint $P_t = \mathrm{vol}$ and an endpoints prescription on P_{01}. He obtains a good back an forth between solutions of his problem and Euler equation. In this thesis, we look at an application of Brenier's ideas to viscous fluids.

1.3.2 Brenier-Schrödinger problem

By the Navier-Stokes equation, we refer to the system :

$$\begin{cases} \partial_t v + \nabla_v v - a\Box v + \nabla p = 0, & (t,x) \in [0,1] \times M \\ \mathrm{div}(v) = 0, & (t,x) \in [0,1] \times M \\ \langle v, \nu \rangle = 0, & (t,x) \in [0,1] \times \partial M \\ v(0,\cdot) = v_0, & x \in M \end{cases} \tag{1.3.1}$$

where M is a compact Riemannian manifold with boundary ∂M, ν is a vector field whose restriction to ∂M is the inward pointing vector field and v_0 is a given initial condition. The unknowns are p, the scalar pressure, and v the velocity vector field. The viscosity terms $a\Box$ is given by the Hodge-de Rham Laplacian.

The first generalization of Brenier's ideas to Navier-Stokes equation comes from [1]. In their work, they consider a minimisation problem on manifold without boundary. A solution is a measure of Brownian motion with drift, minimizing a stochastic kinetic energy where the usual velocity, undefined on Brownian paths, is replaced by the drift.

From this initial article, other authors have taken on the question, giving it its actual formulation and its denomination as Brenier-Schrödinger problem or the portmanteaus Brödinger and Bredinger. Among them, we think to [4], [15], [16], [17], the PhD thesis of A. Baradat [14] and L. Nenna [66] and the article [5] which inspired our work.

The stochastic kinetic energies from [1] and [77] can be expressed as the relative entropy with respect to the reversible Brownian measure R. The variational problem becomes an entropy minimisation problem. The class of measure on which the minimisation is considered is the set $\mathcal{P}(\Omega)$ of probability measures on the path space $\Omega = \mathcal{C}^0([0,1], M)$. The problem is formulated as :

$$H(Q|R) \to \min,\ Q \in \mathcal{P}(\Omega),\ [Q_t = \mu_t,\ \forall t \in \mathcal{T}],\ Q_{01} = \pi. \tag{1.3.2}$$

where $\pi \in \mathcal{P}(M^2)$ is the endpoints distribution, $\mathcal{T} \subset [0,1]$ is measurable and $(\mu_t)_{t\in\mathcal{T}}$ is a family in $\mathcal{P}(M)$ indexed by $\mathcal{T}$. This problem is a mix between Brenier problem, minimization of energy under marginal and endpoint constraints, and Schrödinger problem, minimisation of entropy under marginal constraints (see [41]).

Seen as a problem of his own, there are three main questions :

a) Does it admit a unique solution (or even solutions)?

b) What are the characteristics of a solution?

c) Can we obtain numerical simulations of solutions?

The formulation as a strictly convex minimisation problem brings answers to the three of them. In [16], it is proved that if a solution exist, it is unique and there is a criterion of existence, although not very handy. There is also a characterisation of solutions as reciprocal measures. In [5], is proved a criterion of existence in the torus for the incompressible problem ($\forall t \in [0,1], \mu_t = \mathrm{vol}$). In this same article, Girsanov theorem gives a characterisation of solutions as semi-martingale. Numerical approach of the problem are treated in [17] with Sinkhorn algorithm.

Seen as a variational approach of Navier-Stokes equation, the two main questions are the back and forth between solutions:

d) Is a classical solution of Navier-Stokes equation a solution (and in which sense?) of Brenier-Schrödinger problem?

e) Is a solution of Brenier-Schrödinger problem a solution (and in which sense?) of Navier-Stokes equation?

There is an answer to question d) in [4], for the torus $\mathbb{T}^n$ (see Theorem 6.5.3). Remark that their definition of Navier-Stokes equation is slightly different from our : they do not use the Hodge-de Rham Laplacian. Yet it does not have incidence in flat spaces as toruses. The reversed question is treated in [5], for $\mathbb{R}^n$ and $\mathbb{T}^n$, with the introduction of regular solution. They prove that the backward stochastic velocity satisfies the Newton part of Navier-Stokes equation and the current velocity satisfies the continuity equation.

1.3.3 Contributions of this thesis

In our work, we extend the result of [5] to compact manifold with boundary. In our case, the reference measure R is the reversible measure of reflected Brownian motion. We mostly contribute to questions a), b) and e). The question of existence, and of a good criterion for existence is treated in Chapter 9. We extend the result of [5] from $\mathbb{T}^n$ to compact manifold on which transitively acts a group of isometries, for the incompressible problem. We obtain a finite entropy criterion.

Corollary 1.3.1 (Corollary 9.2.2)**.** *The Brenier-Schrödinger problem* (IBS) *admits a unique solution if and only if* $H(\pi|R_{01}) < \infty$.

Then, we develop a method to obtain existence result on compact manifold with boundary (or with corners) by quotient. It is applied to the segment $[0,1]$, rectangular boxes of n-dimension, regular triangles or n-ball. We finish this study of existence by a problem in a non compact space, $\mathbb{R}^n$:

$$H(P|R) \to \min; [P_t = \mathcal{N}(0, 1/4\,\mathrm{id}), \forall 0 \leq t \leq 1], P_{01} = \pi.$$

We obtain the same criterion (Corollary 9.4.2) although the arguments are slightly different.

The question of characterisation have been deeply resolved by the previous authors yet. In Chapter 7, we re-prove the semi-martingale characterisation in our case. It enlightens the link with variational problem in the manifold setting.

In Chapter 8, we come up to question e) with the stochastic velocity approach of [5]. Our main contribution to this question is the behaviour at the boundary. In Navier-Stokes equation, the impermeability condition states that the velocity is tangent to the boundary. We show that backward (and also forward) velocity has this property. More precisely, we prove the following result.

Theorem 1.3.2 (Theorem 8.4.2)**.** *For P_0 almost all $y \in M$, the backward stochastic velocity $\overleftarrow{v}^y$ satisfies :*

$$\begin{cases} \left(\partial_t + \nabla_{\overleftarrow{v}^y}\right) \overleftarrow{v}^y = \frac{a}{2}\square \overleftarrow{v}^y - \mathbb{1}_{\mathcal{T}}(t)\nabla(ap), & 0 \leq t < 1, t \notin \mathcal{S}, z \in M, \\ \overleftarrow{v}^y_t - \overleftarrow{v}^y_{t^-} = \theta_t(.), & t \in S, z \in M, \\ \langle \overleftarrow{v}^y, \nu(z)\rangle = 0, & z \in \partial M, \\ \overleftarrow{v}^y_0 = -\nabla\eta(., y), & z \in M. \end{cases}$$

Furthermore, there exists a scalar potential φ^y satisfying a second order Hamilton-Jacobi equation, such that

$$\overleftarrow{v}^P_t (X) = -a\nabla\varphi_t^{X_1}(X_t), \; P\text{-}a.s.$$

We also prove the continuity equation again for the current velocity in Theorem 8.5.1.

Chapter 2

Preliminary concepts

The goal of this chapter is to recall the main notions and properties of Riemannian geometry and stochastic calculus on manifolds and fix some notations.

Contents

2.1 Riemannian manifold

In this first section, we recall the notions of Riemannian manifolds, connections and characteristic tensors. As reference, see [30], [46], [50] or [69].

2.1.1 Connection

Let M be a smooth manifold of dimension n. We denote by $\mathcal{C}^\infty(M)$ the space of smooth function from M to $\mathbb{R}$. The tangent space at $x \in M$ is denoted by T_xM and the tangent bundle by TM. A vector field on M is a smooth section of TM. The set of vector field is denoted by $\Gamma(TM)$. For $(x_i)_{1\leq i\leq n}$ a local chart on an open set U, the vector fields $D_i = \partial_{x_i}$ span T_xM for all $x \in U$. As vector spaces, tangent spaces have dual, denoted T_x^*M for $x \in M$. A 1-form is an element of $\Gamma(T^*M)$, smooth section of T^*M.

Unlike in vector spaces, there is not any natural way to add or subtract two elements of TM. For that, we need a convention, a choice : a connection. The easier way to define a connection is as a map $\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ satisfying the following properties : for $X, Y, Z \in \Gamma(TM)$ and $f, g \in \mathcal{C}^\infty(M)$,

$$\begin{aligned} &\nabla_{fX+gY}Z = f\nabla_X Z + g\nabla_Y Z \\ &\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z \\ &\nabla_X fY = f\nabla_X Y + X.fY \end{aligned} \tag{2.1.1}$$

Defined as above, a connection is a way of differentiating vector field. For two vector fields $X, Y \in \Gamma(TM)$, $\nabla_X Y$ is the derivative of Y in the direction X. In the local chart $(x_i)_i$, the connection is characterised by its Christoffel symbols Γ_{ij}^k defined by :

$$\nabla_{D_i} D_j = \sum_{k=1}^n \Gamma_{ij}^k D_k, \ \forall 1 \leq i, j \leq n. \tag{2.1.2}$$

Given a curve γ on M and a connection ∇, a vector field X is said parallel along γ if at every point of γ, we have : $\nabla_{\dot\gamma} X = 0$. We say that the covariant derivative of X along γ vanishes. In this case, X is said to be the parallel translation of $X(\gamma(0))$ along γ. Locally, the parallel translation is uniquely defined. Then, a connection defines a notion of transporting vectors along $\mathcal{C}^1$ curves. A connection also define a notion of straight lines in M : its geodesics. A curve is a geodesic if its velocity is parallel.

A (r,s)-tensor at $x \in M$ is an element of $T_xM^{\otimes r} \otimes T_x^*M^{\otimes s}$. The bundle of (r,s)-tensor is denote by $T^{r,s}M$. A connection can be extended naturally as an application from (r,s)-tensor fields to $(r,s+1)$-tensor fields as a derivation which commute with contraction : for all tensor θ, ψ, and $X \in \Gamma(TM)$

$$\nabla_X(\theta \otimes \psi) = (\nabla_x \theta) \otimes \psi + \theta \otimes (\nabla_X \psi)$$

and for tensor fields in dual bundles,

$$\nabla_X \langle \theta, \psi \rangle = \langle \nabla_X \theta, \psi \rangle + \langle \theta, \nabla_X \psi \rangle.$$

A very common example is the Hessian tensor $\nabla^2 f$ of $f \in \mathcal{C}^2(M)$ defined as the covariant derivative of the differential form df : for all $X, Y \in \Gamma(TM)$ we have

$$\nabla^2 f(X,Y) = \langle \nabla_X df, Y \rangle = X.Y.f - \langle df, \nabla_X Y \rangle.$$

2.1.2 Torsion

To a connection ∇ is attached several tensor fields which describe its properties. The first one is the torsion T. It is the $(1,2)$-tensor defined by :

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y], \ \forall X, Y \in \Gamma(TM), \tag{2.1.3}$$

where $[\cdot,\cdot]$ denotes the Lie bracket. The torsion characterises the symmetry of a connection. A connection is said torsion-free or symmetric if T vanishes. It this case, Christoffel symbol is symmetric :

$$\Gamma_{ij}^k = \Gamma_{ji}^k, \ \forall i, j, k \tag{2.1.4}$$

The torsion characterises the lack of symmetry of the Hessian tensor : for all function $f \in \mathcal{C}^2(M)$ and $X, Y \in \Gamma(TM)$, we have

$$\nabla^2 f(X,Y) = \nabla^2 f(Y,X) + \langle df, T(X,Y) \rangle. \tag{2.1.5}$$

We finish with an interpretation, more geometric, of the torsion, from [69]. The torsion control the closing of small parallelogram. Let $x \in M$ and $A, B \in T_xM$. Given an arbitrary chart $(x_i)_i$ on U, an open neighbourhood of x, there exists a vector field X on U such that $X(x) = A$ and the coordinate of X is the chart (D_i) are constant, equal to the coordinates of A. For t sufficiently small, we can define the integral curve u of X starting from (x, A) and the vector field along u, B_t by parallel translation. For a fixed time t, we can iterate the procedure : there exists a vector field Y on U which coincides with B_t at $u(t)$ and with constant coordinates. We can define v the integral curve of Y starting from $u(t)$. We denote $p_t = v(t)$.

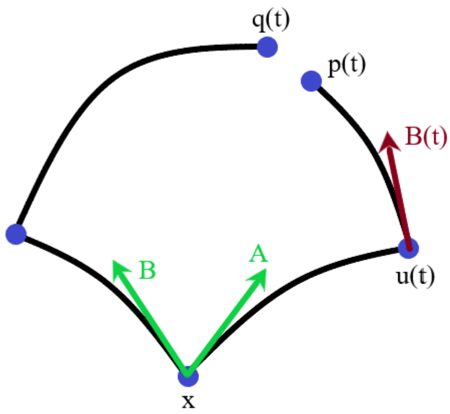


Figure 2.1: Construction of a parallelogram

The same construction, switching the role of A and B, leads to the construction of a point q_t. See Figure 2.1.

In the familiar Euclidean space, we have solely drawn a closed parallelogram. But, in a manifold, p_t and q_t may not be equal. Actually, it is possible to compute the coordinates of these points in our chart and the difference between the i-th coordinate is : $T(A,B)_i t^2 + O(t^3)$. Then, the torsion control closing of parallelograms. With a symmetric connection, parallelograms a closed up to third order.

2.1.3 Metric and Levi-Civita connection

Tangent spaces are merely vector spaces of finite dimension. So as to deal with norm or orthonormal basis, we need to endow them with an inner-product. A metric on M is a smooth family of inner-products g_x on T_xM for all $x \in M$. In other terms, it is a $(0,2)$-tensor field. A manifold M with a metric g is a Riemannian manifold. A metric allows to define a volume measure vol, on M. In a local chart, we have : $\mathrm{vol}(dx) = \sqrt{\det(g)}dx$, where dx is the Lebesgue measure.

A connection is compatible with the metric if $\nabla g = 0$.

Theorem 2.1.1. *On a Riemannian manifold (M,g), there exists a unique torsion-free connection compatible with g.*

The proof of this theorem consists of showing that such a connection satisfies the Koszul formula : for all $X, Y, Z \in \Gamma(TM)$

$$\begin{aligned} 2g(\nabla_X Y, Z) = X.g(Y,Z) + Y.g(X,Z) - Z.g(X,Y) \\ + g([X,Y],Z) - g([X,Z],Y) - g([Y,Z],X). \end{aligned} \tag{2.1.6}$$

This connection is called the Levi-Civita connection, or Riemannian connection.

Let $\gamma : [a, b] \to M$ be a smooth curve, its length is defined by

$$L(\gamma) = \int_a^b |\dot{\gamma}_s| \, dt$$

and its kinetic energy by

$$E(\gamma) = \frac{1}{2} \int |\dot{\gamma}_t|^2 dt$$

The geodesics of the Levi-civita connection are exactly the critical points of E and, locally, curves of minimal length L.

The Laplace-Beltrami operator Δ, or more shortly Laplacian, of a function f is the trace of the Hessian relative to the Levi-Civita connection. If $(X_i)_i$ is an orthonormal basis of T_xM, we have :

$$\Delta f = \sum_i \nabla^2 f(X_i, X_i).$$

2.1.4 Exponential and normal coordinates

Given $x \in M$ and $v \in T_xM$, there exists a unique geodesic γ starting from the initial condition (x, v), defined on some open interval I. For $t \in I$, we define the exponential at x of tv by

$$\exp_x(tv) = \gamma_t.$$

From the properties of geodesics, we know that the map $\exp_x$ is defined on some neighbourhood N_x of $0 \in T_xM$.

Proposition 2.1.2 ([50])**.** *For every $x \in M$, $\exp_x$ is a diffeomorphism from a neighbourhood N_x of $0 \in T_xM$ to a neighbourhood U_x of $x \in M$.*

The injectivity radius at x, $i_M(x)$, is the biggest radius such that the ball centered in x with radius $i_M(x)$ is included in N_x. The injectivity radius i_M is the infimum of $i_M(x)$ over $x \in M$. Considering the union over $x \in M$ of all this $\exp_x$ and identifying M as a sub-manifold of TM, we can define exp from a neighbourhood of M in TM to TM, although i_M is not necessarily positive. The exponential allows us to define a system of coordinates on M. Let (e_i) be a basis of T_xM, the linear isomorphism $(x_i)_i \in \mathbb{R}^n \mapsto \sum_i x_i e_i \in T_xM$ and the exponential map determine an unique system of coordinates : $(x_i)_i \mapsto \exp_x \left(\sum_i x_i e_i\right)$. It is the normal coordinates system in x associated to $(e_i)_i$. In this system, the coordinates of the geodesic γ starting from $(x, \sum v_i e_i)$ are $x_i(\gamma_t) = v_i t$ for all i. The main interest of this system is the vanishing of the Christoffel symbols at x.

Proposition 2.1.3 ([46])**.** *Let $\{x_i\}$ a normal coordinates system at $x \in M$ and Γ the Christoffel symbols of the Levi-Civita connection then :*

$$\Gamma_{ij}^k(x) = 0, \, \forall i, j, k.$$

Remark that if we work with a different connection than the Riemannian one, the Christoffel symbols are only anti-symmetric at x and if the connection is also symmetric (as the Levi-Civita's one), we obtain the same result.

As a local diffeomorphism, $\exp_x$ admits an inverse. For $y \in U_x$, we denote by $\log_x(y)$ the unique vector $v \in N_x$ such that the geodesic γ, starting from (x, v), satisfies $\gamma_1 = y$. This explains the usual notation for $\log_x(y)$ as $\overrightarrow{xy}$.

Proposition 2.1.4. *For all $x \in M$, we have :*

$$d \log_x(x) = \mathrm{id}_{T_x M} \quad \textit{and} \quad \nabla^2 \log_x(x) = 0.$$

2.1.5 Curvature and Ricci tensor

The second characterising tensor is the curvature tensor. It is the $(1, 3)$-tensor defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \, \forall X, Y, Z \in \Gamma(TM). \tag{2.1.7}$$

This object is significantly complex and does not have elegant interpretations as torsion. A convenient way to deal with big tensors as R is to contract them. This is how is constructed the most important tensor in the following work : the Ricci curvature tensor Ric. For $X, Y \in \Gamma(TM)$, $\mathrm{Ric}(X,Y)$ is defined as the trace of the operator $Z \mapsto R(Z,X)Y$. In a Riemannian manifold, the Ricci tensor has the following expression :

$$Ric(X,Y) = \sum_i g(R(X, X_i)X_i, Y), \tag{2.1.8}$$

where $(X_i)_i$ is any orthonormal basis of $T_x M$. This tensor has several interpretation. Locally, it is linked to the volume measure. In a normal coordinates chart $(x_i)_i$, we have :

$$\mathrm{vol}(dx) = \left(1 - \frac{1}{6} \mathrm{Ric}_{ij} \, x_i x_j + O(|x|^3)\right) dx. \tag{2.1.9}$$

This means that in positive Ricci curvature spaces, as $\mathbb{S}^n$, Riemannian balls are smaller than Euclidean balls and in negative Ricci curvature spaces, as $\mathbb{H}^n$, Riemannian balls are bigger. Globally, if Ric is bounded from below by a positive constant, the Bonnet-Myer theorem states that M is compact (with a bound on the diameter depending on Ric). Finally, Ric is linked to the spectrum of Laplacian via Weitzenböck formula. This last interpretation will be deeply studied in Part II.

2.2 The case of Lie groups

Interesting examples of manifolds are the Lie groups. In this section, we recall some facts on Lie group and Lie algebra and connection on Lie groups. For references, see [26] and [69]. This material will be mostly used in Chapter 5.

A Lie group is a group G endowed with a structure of manifold with compatibility between those structures : the application $(x, y) \in G^2 \mapsto xy^{-1}$ is smooth. The common examples of Lie groups are Lie groups of matrices : $GL_n(\mathbb{R})$, and its closed sub-groups. Lie groups are parallelizable : there exists a family of vector fields (X_i) such that for all $g \in G$, $(X(g)$ is a basis of T_gG. In other words, the tangent bundle is trivial. The simplest way to prove it, is to construct left-invariant vector fields. For $g \in G$ the left multiplication $L_g : x \in G \mapsto gx$ is an isomorphism. Its differential acts on vectors. For $v \in T_eG$, tangent space at the identity element e, dL_gv is a vector in T_gG. The left-invariant vector field L_v is defined as $L_v(g) = dL_gv$ for all $g \in G$. If (v_i) is a basis of T_eG, then (L_{v_i}) parallelize TG. The property of parallelizability is somehow a weaker version of the existence of global chart.

To a Lie group G is associated a Lie algebra $\mathfrak{G}$ defined as its tangent space T_eG. The Lie bracket on $\mathfrak{G}$ is the Lie bracket of left-invariant vector in T_eG. Seen as a vector space, $\mathfrak{G}$ can be endowed with an inner-product $\langle \cdot, \cdot \rangle$ which can be extended to the whole tangent bundle by the formula :

$$\langle X, Y \rangle_g = \langle dL_g^{-1}X, dL_g^{-1}Y \rangle, \ \forall g \in G, \forall X, Y \in T_gG. \tag{2.2.1}$$

This metric is left-invariant : left-multiplications are isometric isomorphisms. It endows G with a structure of Riemannian manifold. To this structure is associated a unique Levi-Civita connection whom expression is given by Koszul formula. Yet, this connection has a little problem : its geodesics and the one-parameter subgroups are not necessarily the same curves. There, Riemmanian and group structures show discrepancies. There are connections for which, these curves coincide : the Cartan connections. Among them the left-connection ∇^L and the right-connection ∇^R. They are defined on left-invariant vector fields by

$$\nabla^L_X Y = 0 \quad \text{and} \quad \nabla^R_X Y = [X, Y]. \tag{2.2.2}$$

Left-connection is obviously compatible to the metric, then it cannot be torsion-free. Right-connection has also some torsion. For all left-invariant X, Y, we have :

$$T^L(X, Y) = -[X, Y] = -T^R(X, Y), \tag{2.2.3}$$

Notice that the covariant derivatives of their torsions ∇T vanish. Their curvature tensors also vanish. These connections, ∇^L and ∇^R, satisfy an interesting commutation formula.

Proposition 2.2.1. *For all $X, Y \in \Gamma(TM)$, we have :*

$$\nabla^R_X Y + T^R(X,Y) = \nabla^L_Y X.$$

Proof. Let X, Y be two left-invariant vector fields and f, g two smooth functions. We have :

$$\begin{aligned} \nabla^R_{fX} gY + T^R(fX, gY) &= fg\nabla^R_X Y + f(X.g)Y + fgT^R(X,Y) \\ &\overset{(a)}{=} fg(\nabla^R_X Y + T^R(X,Y)) + f(X.g)Y \\ &= f(X.g)Y \\ &= \nabla^L_{fX} gY \end{aligned}$$

where in (a), we use the formula (2.2.3), valid uniquely for left-invariant vector fields. The $\mathbb{R}$-bilinearity ends the proof. $\square$

2.3 Semi-martingale on a manifold

In this section, we introduce the notion of semi-martingale in a manifold and the two calculus, Ito and Stratonovich, and the particular case of diffusions. See references in [36] and [46].

2.3.1 Semi-martingale

In a vectorial space as $\mathbb{R}^n$, a semi-martingale X of a filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})$ is a continuous adapted process admitting a decomposition as a sum of local martingale and process with finite variations, called Doob-Meyer decomposition. Semi-martingales are "good integrators" : it is possible to define a notion of integral against them. A semi-martingale on $\mathbb{R}^n$ satisfies the Ito formula : for all $\mathcal{C}^2$ function f, we have :

$$f(X_t) = f(X_0) + \int_0^t \langle df(X_s), dX_s\rangle + \frac{1}{2}\int_0^t \nabla^2 f(dX_s, dX_s).$$

This formula can be interpreted as the stability of the class of semi-martingales under $\mathcal{C}^2$ functions. The definition of semi-martingale as a sum highly depends on the additive structure of the space and cannot be extended to general manifolds. There, the notion of semi-martingale will come up from the idea of stability under smooth functions.

Definition 2.3.1. *A semi-martingale is a process X such that for all $f \in \mathcal{C}^\infty(M)$, the real process $f \circ X$ is a semi-martingale.*

The class of "test functions" can be restricted to compactly supported smooth functions $\mathcal{C}_c^\infty(M)$. If (x_i) is a chart, then X is a semi-martingale if and only if its coordinates $X^i = x_i \circ X$ are semi-martingales. This definition "by duality" of semi-martingales is very important. It will be adapted to define martingales, solutions of stochastic differential equations or diffusions. Semi-martingales do not have finite variations but they admit a quadratic variation. It can be seen as a linear map from bilinear forms b to a processes. The b-quadratic variation is denoted :

$$\int b(dX, dX).$$

In a local chart, it has a closed expression. Let $(x_i)_i$ be a local chart, $D_i = \partial_{x_i}$ the moving basis of TM, dx_i the dual basis and X^i the coordinate of the semi-martingale X. If $b = b_{ij} dx_i \otimes dx_j$, the b-quadratic variation is defined by

$$\int_0^t b(dX, dX) = \int_0^t b_{ij}(X_s) d\langle X^i, X^j \rangle_s. \tag{2.3.1}$$

We will see a more intrinsic definition in Chapter 7 when we deal with martingale problems. Now we have semi-martingales, we want to do some calculus as stochastic integration and stochastic differential equation.

2.3.2 Ito differential

We assume that M is endowed with a connection ∇. Let X be a M-valued semi-martingale. The Ito differential of X, denoted $d^\nabla X$ is, formally, an infinitesimal vector. In a chart (x_i), it is defined as :

$$d^\nabla X = \left(dX^i + \frac{1}{2} \Gamma^i_{jk} d\langle X^j, X^k \rangle \right) D_i. \tag{2.3.2}$$

This differential extends the notion of Ito stochastic integral in $\mathbb{R}^n$. It satisfies the generalized Ito formula : for all $f \in \mathcal{C}^2(M)$:

$$f(X_t) - f(X_0) = \int_0^t \langle df, d^\nabla X_s \rangle + \frac{1}{2} \int_0^t \nabla^2 f(dX, dX).$$

As the coordinates X^i are real semi-martingales, they admit a Doob-Meyer decomposition $X^i = M^i + A^i$. The infinitesimal vector $d_m X = dM^i D_i$ is the martingale part of the Ito differential. The advantage of Ito integral is the preservation of martingales. A martingale in a manifold endowed with a connection is a process such that for all function $f \in \mathcal{C}^\infty(M)$, we have

$$f(X(t) \overset{(m)}{=} f(X_0) - \frac{1}{2} \int_0^t \nabla^2 f(dX_s, dX_s)$$

where $\overset{(m)}{=}$ means "up to a local martingale". Then, Ito integral against a martingale is a martingale.

2.3.3 Stratonovich differential

As in the vectorial case, there is an other way to define the stochastic integral, more adapted to calculation : the Stratonovich integral. The Stratonovich differential of X is denoted $\circ dX$. In a local chart, it is given by $\circ dX = \circ dX^i D_i$, where $\circ dX^i$ is the Stratonovich differential of the real semi-martingale X^i. It follows that Stratonovich calculus respect the chain rule : for all $f \in \mathcal{C}^\infty(M)$:

$$f(X_t) = f(X_0) + \int_0^t \langle df, \circ dX \rangle.$$

Yet, the drawback of Stratonovich caluculus is that integral against martingales are no longer martingales. Remark that Stratonovich calculus is defined in a manifold without a connection structure. More than the chain rule, the major interest of Stratonovich calculus is the Stratonovich transfer principle resume by Emery in [38] as : *"geometric constructions performed on differential curves extend intrinsically to random curves by replacing ordinary differentials with Stratonovich stochastic differentials"*. See [37] for more details.

2.3.4 SDE

The Stratonovich calculus is useful to defined stochastic differential equations (SDE) on manifold. Let $(V_i)_{1\leq i\leq l}$ smooth vector fields, Z a semi-martingale in R^l and ξ a random variable on M. A process X is a solution of the SDE

$$\circ dX_t = V_i \circ dZ_t^i,$$

up to a stopping time τ, if for all $f \in \mathcal{C}^\infty(M)$, we have :

$$f(X_t) = f(\xi) + \int_0^t \langle df, V_i \rangle \circ dZ_s^i, \ \forall 0 \leq t \leq \tau.$$

Theorem 2.3.2 ([46])**.** *There exists a unique solution of the SDE $\circ dX_t = V_i \circ dZ_t^i$ uo to its explosion time.*

Idea of the proof. The idea is to use the result of existence and uniqueness in $\mathbb{R}^n$. The manifold M is embedded in some $\mathbb{R}^n$. Then we show that the unique solution stays in M by a Grönwall argument on the square distance from M : $f = d_{\mathbb{R}^n}(\cdot, M)^2$. □

2.3.5 Diffusion and its semi-group

An important class of semi-martingale is the class of diffusions. Let L be a second order operator on $\mathcal{C}^\infty(M)$. A L-diffusion is a process X such that for all $f \in$

$\mathcal{C}^\infty(M)$, the semi-martingale

$$M_t^f = f(X_t) - f(X_0) - \int_0^t Lf(X_s)ds, \ \forall 0 \leq t \leq \tau$$

is a local martingale. Under some regularity conditions on L, there are results of existence and uniqueness of the L-diffusion measure.

Theorem 2.3.3 ([46]). *Let L be a smooth second order elliptic operator and μ_0 a probability measure on M. There exists a unique L-diffusion measure with initial distribution ${}_mu_0$.*

The uniqueness property allows to define the semi-group $\mathbf{P}$ associated to L, on bounded continuous functions. For all $f \in \mathcal{C}_b^0(M)$, $t \geq 0$ and $x \in M$, we have :

$$\mathbf{P}_t f(x) = \mathbb{E}\left[f(X_t^x)\mathbb{1}_{t \leq \tau_x}\right], \tag{2.3.3}$$

where $(X_t^x)_t$ is a L-diffusion, starting from x, defined up to an explosion time τ_x. Diffusion processes satisfy the Strong Markov property. It justifies the designation as "semi-group" : for all $s, t \geq 0$, $\mathbf{P}_{t+s} f = \mathbf{P}_t(\mathbf{P}_s f)$. It also prove the heat-like equation : for all $f \in \mathcal{C}_c^\infty(M)$, we have

$$\partial_t \mathbf{P}_t f(x) = L\mathbf{P}_t f(x) = \mathbf{P}_t L f(x). \tag{2.3.4}$$

2.3.6 Explosion time

As we explained, solutions of SDE or diffusions can be defined up to an explosion time. We present here a criterion of non-explosion for diffusion. We consider a L-diffusion X, starting from $x \in M$, defined up to an explosion time τ_x.

Proposition 2.3.4. *If there exists a smooth function $F : M \to \mathbb{R}$, non negative, such that $\lim_{d(x,y) \to +\infty} F(y) = +\infty$ and a, $c \in \mathbb{R}$ such that $LF \leq cF + a$, then $\tau_x = +\infty$ a.s.*

Proof. Without any loss of generality, we can assume that $a = 0$. First we define the auxiliary semi-martingale $Y_t = e^{-ct}F(X_t)$. According to Ito formula, for all $t \geq 0$, we have :

$$\begin{aligned} Y_t &= F(x) - c\int_0^t e^{-cs}F(X_s)ds + M_t + \int_0 LF(X_s)ds \\ &\leq F(x) + M_t \end{aligned}$$

where M_t is a local martingale. As Y is non negative, with Fatou lemma, it proves that $Y_t \in \mathbb{L}^1$ for all $t \geq 0$. Now, we are going to show that $(Y_t \mathbb{1}_{t\leq\tau_x})_{t\geq 0}$ is a supermartingale. For all $0 \leq s, t$, we have :

$$\begin{aligned}
\mathbb{E}[Y_{t+s}\mathbb{1}_{t+s\leq\tau_x}|\mathcal{F}_s] &= e^{-c(t+s)}\mathbb{E}[F(X_{t+s})\mathbb{1}_{t+s\leq\tau_x}|\mathcal{F}_s] \\
&= e^{-c(t+s)}\mathbb{E}_{X_s}[F(X_t)\mathbb{1}_{t\leq\tau}]\mathbb{1}_{s\leq\tau_x} \\
&= e^{-cs}e^{-ct}\mathbf{P}_t F(X_s)\mathbb{1}_{s\leq\tau_x}
\end{aligned}$$

where $\mathbf{P}$ is the semi-group associated to the L-diffusion. Then for all $y \in M$, we have :

$$\mathbf{P}_t F(y) = F(y) + \int_0^t L\mathbf{P}_s F(y) ds. \tag{2.3.5}$$

According to Grönwall lemma, we have : $\mathbf{P}_t F(y) \leq F(y)e^{ct}$ a.s. Then, $(Y_t\mathbb{1}_{t\leq\tau_x})_{t\geq 0}$ is a supermartingale. Now, we defined the following family of stopping times : for all $n \in \mathbb{N}^*$

$$T_n =\in \{t \geq 0 : F(X_t) \geq n\}. \tag{2.3.6}$$

By definition, $(T_n)_{n\geq 0}$ is increasing and converge to τ_x. For all $n \geq 0$, we have :

$$\begin{aligned}
n\mathbb{P}(t > T_n) &= \mathbb{E}[F(X_{T_n})\mathbb{1}_{t>T_n}] \\
&\leq \mathbb{E}[F(X_{t\wedge T_n})] \\
&\leq e^{ct}\mathbb{E}[Y_{t\wedge T_n}\mathbb{1}_{t\wedge T_n\leq\tau_x}] \\
&\leq e^{ct}F(x)
\end{aligned}$$

Then, for all $t \geq 0$, $\mathbb{P}(t > T_n) \to 0$ as $n \to +\infty$. By monotonous convergence, for all $t \geq 0$, $\mathbb{P}(t > \tau_x) = 0$. □

2.4 Horizontal lift

In this section, we construct the parallel translation $/\!/$ over a semi-martingale. When the semi-martingale is a diffusion process, we recall the diffusion properties of $/\!/$. For more details on this construction, see references in [46].

2.4.1 Connection (bis)

Definition 2.1.1 is handy for calculation but not very meaningful. We make it clearer with the notion of horizontal vector. In a manifold M endowed with a connection ∇, let $x \in M$ and $v \in T_xM$. Among the curve in TM starting from v, we call vertical those which stay in T_xM. The tangent vector of a vertical curve in v is called a vertical vector. We denote by V_vTM the subspace of T_vTM of vertical vector field. A curve in TM is horizontal if it is parallel to its projection in M. We denote by H_vTM the space of horizontal vectors in T_vTM, tangent to horizontal curves. These two vector spaces are supplementaries :

$$V_vTM \oplus H_vTM = T_vTM.$$

The space V_vTM is canonical. The choice of a connection ∇ is equivalent to the choice of a supplementary to V_vTM for all $v \in TM$.

This decomposition extends to the frame bundle $\mathcal{F}(M)$. A frame above $x \in M$ is an isomorphism $u : \mathbb{R}^n \to T_xM$. The fibres, $\mathcal{F}(M)_x$, are isomorphic to $GL_n(\mathbb{R})$. For u in $\mathcal{F}(M)$, we denote $\pi(u)$ the projection in M. The tangent space $T_u\mathcal{F}(M)$ admits a decomposition as vertical and horizontal vectors. A vector is vertical if it is tangent to the fibre $\mathcal{F}(M)_{\pi(u)}$. A curve u_t in $\mathcal{F}(M)$ is horizontal if for all $e \in \mathbb{R}^n$, the vector field u_te in TM is parallel along the curve $\pi(u_t)$ in M. A vector in $T_u\mathcal{F}(M)$ is horizontal if it is tangent to an horizontal curve. We denote by $H_u\mathcal{F}(M)$ the space of horizontal vector. We have :

$$T_u\mathcal{F}(M) = V_u\mathcal{F}(M) \oplus H_u\mathcal{F}(M).$$

An argument of dimension shows that $H_u\mathcal{F}(M)$ is isomorphic to $T_{\pi(u)}M$. We denote by π_* the isomorphism induced by π and h_u its reciprocal : for all $v \in T_{\pi(u)}M$, the unique vector in $H_u\mathcal{F}(M)$ such that $\pi_*(H_u(v)) = v$ is denoted by $h_u(v)$. Let $(e_i)_i$ be the canonical basis of $\mathbb{R}^n$, we defined a basis of $H_u\mathcal{F}(M)$ for all frame $u \in \mathcal{F}(M)$ by :

$$H_i(u) = h_u(ue_i),\ \forall 1 \leq i \leq n.$$

2.4.2 Horizontal lift - $\mathcal{C}^1$ case

A curve in a manifold is, by nature, a complex object. The idea of horizontal lift is to summarize a curve in M in a curve in $\mathbb{R}^n$. Given a curve $\gamma \in M$ and a frame $u_0 \in \mathcal{F}(M)_{\gamma(0)}$, there exists a unique parallel curve u in $\mathcal{F}(M)$ such that $\pi(u(t)) = \gamma(t)$ and $u(0) = u_0$. This curve is called the horizontal lift of γ starting from u_0. Uniqueness and existence come from the resolution of differential equation satisfied by u. The application $u_t \circ u_0^{-1} : T_{\gamma(0)}M \to T_{\gamma(t)}M$ is the parallel translation along γ as defined in Section 2.1.1. In particular, it does not depend on the choice of u_0. The anti-development of γ (or of u), is the vectorial curve w defined by :

$$\dot{w}_t = u_t^{-1}\dot{\gamma}_t,\ w_0 = 0.$$

Rewriting this equation, we have :

$$\dot{u}_t = H_i(u_t)\dot{w}_t^i. \tag{2.4.1}$$

In the case of Riemannian manifold endowed with its Levi-Civita connection, it become more relevant to restrict our study on the orthonormal frames bundle $\mathcal{O}(M)$. If $u \in \mathcal{O}(M)$ the vectors $H_i(u)$ are tangent to $\mathcal{O}(M)$ and form an orthonormal basis. A major property of parallel translation in the Riemannian case is its respect of the metric.

Proposition 2.4.1. *Let $\gamma : I \to M$ be a $\mathcal{C}^1$ curve. If X_t and Y_t are vector fields parallel along γ for the levi-Civita connection, then :*

$$\langle X_t, Y_t \rangle = \langle X_0, Y_0 \rangle, \ \forall t \in I.$$

Proof.

$$\begin{aligned} \frac{d}{dt}\langle X_t, Y_t\rangle &= \nabla_{\dot{\gamma}_t}\langle X_t, Y_t\rangle \\ &= \langle \nabla_{\dot{\gamma}_t} X_t, Y_t\rangle + \langle X_t, \nabla_{\dot{\gamma}_t} Y_t\rangle \\ &= 0 \end{aligned}$$

□

Remark that this calculation stays true for any connection adapted to the metric.

2.4.3 Horizontal lift - Stochastic case

The construction of horizontal lift and anti-development of a semi-martingale in M is an example of Stratonovich transfer principle application. An horizontal semi-martingale in $\mathcal{F}(M)$ is a solution to the SDE

$$\circ dU_t = H_\alpha(U_t) \circ dW_t^\alpha \tag{2.4.2}$$

where W is a semi-martingale in $\mathbb{R}^n$, called the anti-development of U in $\mathbb{R}^n$. The semi-martingale U is called the development of W in $\mathcal{F}(M)$ and the horizontal lift of its projection $X = \pi(U)$. X is the development of W in M. This construction can be traced back, in an uncanonical way though, if M is embedded in some $\mathbb{R}^N$. We denote $P(x) : \mathbb{R}^N \to T_xM$ the orthogonal projection and P^* its horizontal lift, we have :

$$\begin{tikzcd} Z \arrow[d, bend left, "\circ dU_t = H_\alpha(U_t)\circ dZ_t^\alpha"] \\ U \arrow[u, bend left, "\circ dZ_t = U^{-1}P_\alpha(X_t)\circ dX_t^\alpha"] \arrow[d, bend left, "X=\pi(U)"] \\ X \arrow[u, bend left, "\circ dU_t = P^*_\alpha(U_t)\circ dX_t^\alpha"] \end{tikzcd}$$

This shows that once a starting frame U_0 is fixed, there is a one-to-one correspondence $X \leftrightarrow U \leftrightarrow W$. Remark that, as in the smooth case, if U_0 is in $\mathcal{O}(M)$, U stays in $\mathcal{O}(M)$ and keeps preserving the metric. Horizontal lift and anti-development keep the properties of the semi-martingale X. For example, X is a martingale if and only if W is a local martingale. This is the case for diffusion properties. Let $L = \Delta + b$ be a second order operator, with $b \in \Gamma(TM)$ and X a diffusion with generator L. Let U its horizontal lift with starting frame $U_0 \in \mathcal{O}(M)$. Then U is a diffusion on $\mathcal{O}(M)$. Its generator is $L^{\mathcal{O}} = \Delta_{\mathcal{O}} + \nabla_b$, where $\Delta_{\mathcal{O}}$ is the Bochner Laplacian defined as :

$$\Delta_{\mathcal{O}} = \sum_i H_i^2.$$

This Laplacian is intertwined to the Laplace-Beltrami Laplacian by the projection π :

$$\Delta_M(f \circ \pi) = (\Delta_{\mathcal{O}} f) \circ \pi.$$

The parallel translation above X is defined, as in the smooth case, by :

$$/\!\!/_t = U_t U_0^{-1},\ \forall 0 \leq t \leq \tau.$$

It is an isometric isomorphism between $T_{X_0}M$ and $T_{X_t}M$. If X is a diffusion, $/\!\!/$ can be seen as a diffusion on TM. Its generator on 1-forms is given by :

$$L^{/\!\!/} = \Delta^h + \nabla_b, \tag{2.4.3}$$

where Δ^h is the horizontal Laplacian on 1-form :

$$\Delta^h \alpha = \sum_i \nabla^2 \alpha(X_i, X_i),\ \forall \alpha \in \Gamma(T^*M),$$

with $(X_i)_i$ any orthonormal basis.

2.5 Interlude - On "The" Laplacian on $\Gamma(T^*M)$

In this section we discus the notion of Laplacian on 1-forms and we compare the two classical Lapalcians : horizontal and Hodge-de Rham. The key Weitzenböck formula from Theorem 2.5.1. See [46] for references.

The horizontal Laplacian Δ^h defined above is a natural extension of Laplace-Beltrami operator. It is the trace of the Hessian ∇^2. Yet, this operator does not have good geometric properties and is often leaved in favor of the Hodge-de Rham Laplacian. This Laplacian has a Hilbertian construction. Let d the exterior differentiation on $\Lambda(M)$ and δ its adjoint for the inner-product

$$(\alpha, \beta) = \int_M \langle \alpha, \beta \rangle \, dx.$$

The the Hodge-de Rham Laplacian is defined by :

$$\square = -(d+\delta)^2. \tag{2.5.1}$$

Remark that it coincides with Δ_M on functions (i.e, on 0-forms). Also remark that Hodge-de Rham Laplacian satisfies the fundamental commutation formula :

$$d\Delta_M = \square d. \tag{2.5.2}$$

This operator is linked to the horizontal Laplacian by the Weitzenböck formula.

Theorem 2.5.1 (Weitzenböck formula -[46])**.** *For all $\alpha \in \Gamma(T^*M)$, we have :*

$$\square\alpha = \Delta^h\alpha - \mathrm{Ric}^{\#}\,\alpha,\ \forall\alpha \in \Gamma(T^*M).$$

Idea of the proof. A straightforward proof is the computational one. Let $(x_i)_i$ a normal chart, D_i the associated basis of T_xM and dx_i the dual basis of T^*M. For all $\alpha \in \Gamma(T^*M)$, we show that : $d\alpha = dx_j \wedge \nabla_{D_j}\alpha$ and $\delta\alpha = -i(D_j)\nabla_{D_j}\alpha$ where $i(X)$ is the interior product : $i(X)\theta(\cdot) = \theta(X,\cdot)$. Then, we show :

$$\square_M\alpha = \nabla_{D_j}\nabla_{D_j}\alpha + dx_j \wedge i(D_k)\left(\nabla_{D_j}\nabla_{D_k} - \nabla_{D_k}\nabla_{D_j}\right).$$

We can recognise the terms. $\square$

Part II

Intertwining and functional inequalities

"You see, it is simply a very young girl's record of her own thoughts and impressions, and consequently meant for publication. When it appears in volume form I hope you will order a copy."
OSCAR WILD, *The importance of beeing earnest*

Chapter 3

Deformed parallel translation and intertwining

We define the deformed parallel translation and the intertwining relation at the level of processes, generators and semi-group under Bakry-Émery criterion. It is applied to functional inequalities and concentration result.

Contents

3.1 Introduction

This chapter is an introduction to intertwining and a stochastic study of it. It motivates Bakry-Émery criterion as pivotal point in functional inequalities and prepare our major contribution presented in Chapter 4. There are different types of intertwining. It can be through a function between two manifold $p : N \to M$ as in [61], or it can be through a derivative. To a differential operator on smooth functions, it is possible to associate a diffusion process in one hand and a semi-group acting on function on the other hand. We are looking to the action of the differentiation on this three levels. The goal is to rewrite the derivative of a smooth Markov semi-group acting on functions as a Markov semi-group acting on differential forms. Unlike in the one-dimensional case, where functions and their derivatives have the same nature, in a manifold setting, the two intertwined semi-groups act on different spaces. Actually, we look at semi-groups on 1-forms which restrictions on differential forms satisfy an intertwining relation. As we want to stress on the action on 1-forms, we do not look into Bismut type formulae (see [35] or [34]).

These relations have been first investigated in the discrete case for birth-death processes in [25] and in the one dimensional case in [18]. The case of reversible ergodic diffusions in the Euclidean space $\mathbb{R}^n$ is treated in [2]. In this chapter, we also investigate the case of reversible and ergodic diffusions, on a complete connected Riemannian manifold M, with generator

$$Lf = \Delta f - \langle \nabla V, \nabla f \rangle,$$

where V is a smooth potential on M. Such diffusions admit a unique invariant measure, μ, absolutely continuous with respect to the Riemannian measure, with density proportional to e^{-V}.

At the level of operators, the intertwining relation occurs without further assumptions. The generator L is intertwined with a weighted Laplacian acting on 1-forms, L^W, unitary equivalent to the Witten Laplacian. A large study of this operator can be found in the work of Helffer, with application to correlation decay in spin systems (see [45]). At the level of stochastic processes, L^W is the generator on 1-forms of a diffusion on the tangent bundle: the deformed parallel translation (or geodesic transport in [62]). In [3], this process appears naturally as a spacial derivative of a flow of the diffusion with generator L. These intertwining relations at the level of processes and generator suggest an intertwining relation at the level of the semi-groups and a stochastic representation of the intertwined semi-group. However, at the level of semi-group, this relation is not so obvious: more assumptions are required. In the Euclidean space, the classical assumption is the strong convexity of the potential V or, in other way to say it, the positiveness of its Hessian. A classical generalization of this condition on Riemannian manifolds is the positiveness of an operator depending on Hessian and Ricci curvature, known as the Bakry-Émery criterion (see [8]). The stochastic approach of intertwining relation is an important part of Li's PhD thesis [58], in which many results of this chapter have already been proved, and works ([59], [60]) and the books of Elworthy, Le Jan and Li [32] and [33].

Let us summarize the content of this chapter. In Section 3.2, we define the deformed parallel translation and the intertwining relation at the level of processes and generators. In Section 3.3, we make a link between covariance representation and intertwining at the level of semi-group. It is a motivation to look at intertwining by differentiation. In Section 3.4, we study the commutation of the $\mathcal{C}^0$-semi-group under the Bakry-Émery criterion. We obtain new proofs of some well-known results, using only stochastic tools.

3.2 Deformed parallel transport and Commutation

In this section, we introduce the deformed parallel translation. It is defined as a stochastic process by a covariant differential equation. The goal is to establish intertwining relations at the level of stochastic processes and at the level of generator.

On a connected complete Riemannian manifold $(M, \langle \cdot, \cdot \rangle)$, endowed with its Levi-Civita connection ∇, let $\mathcal{C}^\infty(M)$ be the space of smooth real-valued functions and $\mathcal{C}_c^\infty(M)$ its subspace of compactly supported functions. We consider the second order diffusion operator defined on $\mathcal{C}^\infty(M)$ by

$$Lf = \Delta f - \langle \nabla V, \nabla f \rangle, \tag{3.2.1}$$

where V is a smooth potential. We denote by μ the measure on M with density e^{-V}. On $\mathcal{C}_c^\infty(M)$, the operator L is symmetric with respect to μ, that is for all $f, g \in \mathcal{C}_c^\infty(M)$,

$$\int_M Lf g \, d\mu = -\int_M \langle df, dg \rangle \, d\mu = \int_M f Lg \, d\mu. \tag{3.2.2}$$

Let X_t^x be a diffusion process with generator L, started at $x \in M$. Such a process exists and is unique in law, up to an explosion time τ_x. We define the associated semi-group $(\mathbf{P}_t)_{t \geq 0}$ on the space of bounded continuous functions as in (2.3.3) :

$$\mathbf{P}_t f(x) = \mathbb{E}\left[f(X_t^x) \mathbb{1}_{t \leq \tau_x}\right].$$

Above the process X_t^x, one can construct the parallel translation $/\!\!/_t$. It is an isometric isomorphism from T_xM to $T_{X_t^x}M$ but in a stochastic point of view, it can be seen as a diffusion on the tangent bundle. Its generator on 1-forms is given by :

$$L^{/\!\!/} \alpha = \Delta^h \alpha - \nabla_{\nabla V} \alpha, \ \forall \alpha \in \Gamma(T^*M) \tag{3.2.3}$$

where Δ^h is the horizontal Laplacian on 1-forms. For more details on the construction of this object, see Chapter 2.

The parallel translation is the first step to define a more relevant translation, in terms that will be explained bellow : the deformed parallel translation, or damped parallel translation, W_t. It is the linear map $T_xM \to T_{X_t^x}M$ determined by the differential equation:

$$\begin{cases} D_t W_t v = -\mathcal{M}^* W_t v dt \\ W_0 = \mathrm{id}_{T_xM} \end{cases}, \tag{3.2.4}$$

where

$$D_t W_t v = /\!\!/_t d\left(/\!\!/_t^{-1} W_t v\right) \tag{3.2.5}$$

stands for the covariant derivative of $W_t v$ and the operator $\mathcal{M}^*$ is a section of $\mathrm{End}(TM)$ defined by

$$\mathcal{M}^* w = \nabla_w \nabla V + \mathrm{Ric}(w),\ \forall w \in TM. \tag{3.2.6}$$

Its adjoint operator, a section of $\mathrm{End}(T^*M)$ is denoted $\mathcal{M}$. Remarks that both translations $/\!/_t$ and W_t depend on the initial point x but we avoid any reference to it when it is obvious. As an alternative definition or a major property, Theorem 2.1 in [3] shows that for all $x \in M$ and $v \in T_x M$, $W_t v$ is the spatial derivative of a flow of the diffusion with generator L, obtained from X_t^x by parallel coupling. In some way, the processes X_t^x and W_t are intertwined. As the parallel translation, the deformed parallel translation can be seen as a diffusion on the tangent bundle.

Proposition 3.2.1. *The generator on* 1*-forms of the deformed parallel translation is given by :*

$$L^W \alpha = L^{/\!/} \alpha - \mathcal{M}\alpha,\ \forall \alpha \in \Gamma(T^*M). \tag{3.2.7}$$

Proof. This result is just an application of Ito formula to 1-form (see [58]). Let us detail a bit. Let α be a 1-form and $v \in T_x M$, we have :

$$\begin{aligned} d\langle \alpha, W_t v\rangle &= d\langle \alpha /\!/_t, /\!/_t^{-1} W_t v\rangle \\ &= \left\langle d\left(\alpha /\!/_t\right), /\!/_t^{-1} W_t v\right\rangle + \left\langle \alpha /\!/_t, d\left(/\!/_t^{-1} W_t v\right)\right\rangle + \left\langle d(\alpha /\!/_t), d(/\!/_t^{-1} W_t)\right\rangle \\ &= \langle D_t \alpha, W_t v\rangle + \langle \alpha, D_t W_t v\rangle + \langle D_t \alpha, D_t W_t\rangle, \end{aligned}$$

where $D_t \alpha = d\left(\alpha /\!/_t\right) /\!/_t^{-1}$ stands for the covariant differential of α along X_t^x. As parallel translation is a diffusion with generator $L^{/\!/}$, we have :

$$\langle D_t \alpha, W_t v\rangle \overset{(m)}{=} \langle L^{/\!/} \alpha, W_t v\rangle dt,$$

where $\overset{(m)}{=}$ means "up to a local martingale". As $W_t(x)$ satisfies equation (3.2.4), we obtain the second term and the quadratic term $\langle D_t \alpha, D_t W_t\rangle$ vanishes as $D_t W_t$ has finite variation. □

For now, the operator L^W has been defined only on smooth 1-forms. We extend it in a L^2-sense. Let $L^2(\mu)$ be the space of measurable 1-forms α such that

$$\int_M |\alpha|^2 \, d\mu < +\infty.$$

Following the ideas of [28], Li proved in [58] the following result.

Theorem 3.2.2. *The operator L^W is essentially self-adjoint on $L^2(\mu)$.*

We give some ideas of the proof.

Proof. We denote by δ_V the adjoint of the exterior derivative on forms for the scalar product on $L^2(\mu)$. Some calculation shows that, for all smooth compactly supported 1-forms α, we have:

$$L^W \alpha = -(d\delta_V + \delta_V d)\alpha \tag{3.2.8}$$

Then L^W is unitary equivalent to a Witten Laplacian and so is essentially self-adjoint (see [45] for more details). $\square$

Then, without any assumptions, the deformed parallel translation defines a semi-group $(\mathbf{Q}_t)_{t\geq 0}$ on $L^2(\mu)$. We will see in Section 3.4 that under suitable conditions on the potential $\mathcal{M}$, it also generate a $\mathcal{C}^0$ semi-group, on bounded continuous 1-forms with a stochastic representation as (2.3.3). Remark that a continuous bounded 1-form is not bounded as an element of $\mathcal{C}^0(TM)$ and this is the major obstruction to the definition of a $\mathcal{C}^0$ semi-group.

The generator of the deformed parallel translation satisfies a commutation formula. For all $f \in \mathcal{C}^\infty(M)$, one has:

$$dLf = L^W df. \tag{3.2.9}$$

This is an intertwining relation at the level of generators. This commutation formula on generators and the intertwining relation at the level of stochastic processes suggest an intertwining relation between the semi-groups $\mathbf{P}$ and $\mathbf{Q}$.

3.3 A covariance representation

In this section, we present a well-known integral representation of the covariance Cov_μ and explain our motivation to obtain intertwining relations. We assume that μ is a probability measure. Then, it makes sense to look forward bounds on it variance. We also assume that the diffusion is ergodic i.e for all $f \in \mathcal{C}_c^\infty(M)$:

$$\lim_{t\to+\infty} \mathbf{P}_t f = \mu(f) \text{ a.s.} \tag{3.3.1}$$

This notion of ergodicity is implied by coupling property. Usually, speaking of ergodicity, we refer to a stronger notion, with L^2 convergence which is not necessary here.

Proposition 3.3.1. *For all $f, g \in \mathcal{C}_c^\infty(M)$ we have the following covariance representation:*

$$\mathrm{Cov}_\mu(f,g) = \int_0^{+\infty} \left(\int_M \langle df, d\mathbf{P}_t g\rangle \, d\mu \right) dt.$$

Proof. Using ergodicity and the relation (2.3.4), for all $f, g \in \mathcal{C}_c^\infty(M)$ we have :

$$\begin{aligned}
\mathrm{Cov}_\mu(f,g) &= \int_M f(g - \mu(g))\, d\mu \\
&= \lim_{t\to+\infty} \int_M f(g - \mathbf{P}_t g)\, d\mu \\
&= -\int_M \int_0^{+\infty} f L \mathbf{P}_t g \, dt\, d\mu \\
&= \int_0^{+\infty} \left(\int_M \langle df, d\mathbf{P}_t g\rangle \, d\mu \right) dt
\end{aligned}$$

□

This covariance representation enlightens the necessity of understanding the differential $d\mathbf{P}_t$. We finish with a heuristic calculation. Assume that the intertwining relation is satisfied, we formally have :

$$\begin{aligned}
\mathrm{Var}_\mu(f) &= \int_0^{+\infty} \left(\int_M \langle df, \mathbf{Q}_t df\rangle \, d\mu \right) dt \\
&= \int_M \langle df, (L^W)^{-1} df\rangle \, d\mu \\
&\leq \int_M \langle df, \mathcal{M}^{-1} df\rangle \, d\mu
\end{aligned}$$

The last inequality is the Brascamp-Lieb inequality. This is the kind of functional inequalities we are interested in. This calculation can be rigorously establish under the Bakry-Émery criterion, presented in the next section. It is developed in [12] for instance. This idea will be the core of Chapter 4.

3.4 $\mathcal{C}^0$ semi-group under Bakry-Émery criterion

The main goal of this section is to motivate the use of intertwining relation for functional inequalities and the role of Bakry-Émery Here, we obtain an asymmetric Brascamp-Lieb inequality in the spirit of Ledoux (see [54] or [23]). This inequality is called asymmetric because it gives an L^1-L^∞ bound of the covariance. The Brascamp-Lieb inequality will be treated in Section 4.5 and 4.6 with an L^2 approach of the semi-group. This section is also the opportunity to give a proof of the intertwining relation for the $\mathcal{C}^0$ semi-groups, using only the stochastic tools presented in Section 3.2. Firstly, we have to find a condition so as to properly define the semi-group. As an endomorphism of T_x^*M, $\mathcal{M}(x)$, defined in (3.2.6), is symmetric with respect to the metric. We denote by $\rho(x)$ the smallest eigenvalue of $\mathcal{M}(x)$ and by ρ, its infimum over M:

$$\rho = \inf_{x\in M}\{\text{smallest eigenvalue of}\, \mathcal{M}(x)\} \tag{3.4.1}$$

The key assumption of this section is the Bakry-Émery criterion, also known as $CD(\rho,\infty)$ condition in [12].

Assumption 3.4.1 (Bakry-Émery criterion)**.** *Let us assume that the operator $\mathcal{M}$ is uniformly bounded from below, i.e $\rho > -\infty$.*

It is a sufficient condition for hypercontractivity of the diffusion and allows to prove Poincaré or Log-Sobolev inequalities (see [11]). Bakry proves in [8] that, under this criterion, the diffusion X does not explode (i.e for all $x \in M$, $\tau_x = +\infty$ almost surely). It is not a necessary condition. The following intertwining results are proved in [58] with a finite moment criterion, weaker but less handy, and for other flow (see also the concept of p-completeness in [59]), or in [35] with finer bounds.

Proposition 3.4.2. *Under the Bakry-Émery criterion, the semi-group* $\mathbf{Q}$ *has the stochastic representation : for all bounded* 1*-form* α*, for all* $x \in M$*, for all* $v \in T_vM$*,*

$$\langle \mathbf{Q}_t\alpha, v\rangle = \mathbb{E}\left[\langle \alpha, W_t v\rangle\right], \tag{3.4.2}$$

and we have: for all 1*-form* α*, for all* $t \geq 0$*,*

$$\|\mathbf{Q}_t\alpha\|_\infty \leq e^{-\rho t}\|\alpha\|_\infty.$$

Proof. The heart of the proof is to show that under this criterion, the deformed parallel translation is bounded. For all $x \in M$ and all $v \in T_xM$, one has

$$\begin{aligned} d|W_t v|^2 &= 2\,\langle W_t v, D_t W_t v\rangle \\ &= -2\,\langle W_t v, \mathcal{M}^* W_t v\rangle\, dt \\ &\leq -2\rho|W_t v|^2 dt. \end{aligned}$$

By Grönwall lemma, this yields

$$|W_t v| \leq e^{-\rho t}|v|,\ a.s. \tag{3.4.3}$$

Remark that this bound does not depend on the initial point x. This shows that the stochastic representation (3.4.2) is well-defined and concludes the proof. □

Proposition 3.4.3. *Under the Bakry-Émery criterion, the semi-groups* $\mathbf{P}$ *and* $\mathbf{Q}$ *are intertwined by the derivative of functions: for all* $f \in \mathcal{C}_c^\infty(M)$*,*

$$d\mathbf{P}_t f = \mathbf{Q}_t df. \tag{3.4.4}$$

Proof. Let $x \in M$, $v \in T_xM$ and $\gamma : I \to M$ a smooth curve such that $\gamma(0) = x$ and $\gamma'(0) = v$. According to [3], there exists a flow $X_t(a)$ of the L-diffusion such that $X_t(0) = X_t^x$, $X_0(a) = \gamma(a)$ and $\partial_a X_t(a) = W_t(a)\gamma'(a)$ where $W_t(a)$ is the deformed parallel translation above $X_t(a)$. For any $f \in \mathcal{C}_c^\infty(M)$ and $t > 0$, we have :

$$\begin{aligned}\langle d\mathbf{P}_t f, v\rangle &= \left.\frac{d}{da}\right|_{a=0} \mathbf{P}_t(\gamma(a)) \\ &= \left.\frac{d}{da}\right|_{a=0} \mathbb{E}\left[f(X_t(\gamma(a)))\right]\end{aligned}$$

The bound 3.4.3 and the regularity of f guarantee the differentiation under the expectation. We have :

$$\begin{aligned}\langle d\mathbf{P}_t f, v\rangle &= \mathbb{E}\left[\langle df, W_t v\rangle\right] \\ &= \langle \mathbf{Q}_t df, v\rangle\end{aligned}$$

□

This result has also been proved for q-form in [32]. Armed with it, it is possible to obtain several result in analysis. Amongst them, there are finiteness results of volume and homotopy group (see [60]). We are going to use this intertwining relation to obtain functional inequalities, in the spirit of Chapter 4. We get back to the assumptions of ergodicity and finite measure from Proposition 3.3.1. To begin with, we can rewrite the integral representation of the covariance using the intertwining. For all $f, g \in \mathcal{C}_c^\infty(M)$, we have :

$$\mathrm{Cov}_\mu(f, g) = \int_0^{+\infty} \int_M \langle df, \mathbf{Q}_t(dg)\rangle \, d\mu \, dt. \tag{3.4.5}$$

It is the key argument to prove the following asymmetric Brascamp-Lieb inequality.

Theorem 3.4.4 (Asymmetric Brascamp-Lieb inequality)**.** *Assume that $\rho > 0$, then for all f, $g \in \mathcal{C}_c^\infty(M)$, we have*

$$|\mathrm{Cov}_\mu(f, g)| \leq \frac{1}{\rho} \|dg\|_\infty \int_M |df| \, d\mu.$$

Proof. From Hölder inequality, for all $f, g \in \mathcal{C}_c^\infty(M)$ and $t \geq 0$, we have :

$$\left|\int_M \langle df, \mathbf{Q}_t(dg)\rangle \, d\mu\right| \leq \|\mathbf{Q}_t(dg)\|_\infty \int_M |df| \, d\mu.$$

Using the bound from Proposition 3.4.2, we have :

$$|\mathbf{Q}_t(dg)\|_\infty \leq e^{-\rho t}\|dg\|_\infty.$$

With the representation (3.4.5) of the covariance, it ends the proof. □

Remark that in [8], the assumption of finiteness of μ is proven to be implied by the positivity of ρ.

A consequence of Theorem 3.4.4, is the Gaussian concentration of the probability μ. This concentration result has been shown by Ledoux in [51] for the volume measure of a compact Riemannian manifold under the condition of positive Ricci curvature and in [52] in the Euclidean space under the condition of strictly convex potential. This inequality is deeply exposed in [53]. Our proof gives a new outlook of the result, with only stochastic tools.

Proposition 3.4.5. *If $\rho > 0$, then for all 1-Lipschitz $f \in \mathcal{C}_c^\infty(M)$ and for all $r > 0$,*

$$\mu\left(|f - \mu(f)| > r\right) \leq 2e^{-\rho\frac{r^2}{2}}. \tag{3.4.6}$$

Proof. The idea of the proof is to bound the Laplace transform. Let f be a smooth compactly supported 1-Lipschitz function. Without any loss of generality, we can assume that f is centered. For any $\lambda > 0$, we have:

$$\begin{aligned}\frac{d}{d\lambda}\mathbb{E}_\mu[e^{\lambda f}] &= \mathrm{Cov}_\mu(f, e^{\lambda f})\\ &\leq \frac{1}{\rho}\|df\|_\infty \int_M \lambda|df|e^{\lambda f}\,d\mu\\ &\leq \frac{\lambda}{\rho}\mathbb{E}_\mu\left[e^{\lambda f}\right].\end{aligned}$$

By Grönwall lemma, it yields :

$$\mathbb{E}_\mu[e^{\lambda f}] \leq e^{\lambda^2/2\rho}. \tag{3.4.7}$$

The proof ends by using Markov's inequality and optimizing in λ. □

Chapter 4

Twistings and intertwined semi-groups

So as to obtain intertwining, we defined a twisted semi-group. It is applied to functional inequality as Poincaré and illustrated with examples in radially symmetric surface. This chapter exposes the main results of [47].

Contents

4.1 Introduction

The Bakry-Émery criterion and the strict convexity of $\mathcal{M}$ appeared in Chapter 3 as natural conditions for intertwinings and functional inequalities for the measure μ. Yet, these conditions are quite limiting and unnecessary. For example, the potential $V(x) = x^4/4$ on $\mathbb{R}$ is not strictly convex but the associated measure has a good spectral gap property. In this chapter, we present a method which oversteps this problem. While we considered the $\mathcal{C}^0$-semi-group in Chapter 3, we are now interested in the L^2 one. In this chapter, we extend a strategy presented in [2] on $\mathbb{R}^n$, to manifold, so as to obtain intertwined semi-groups even if the Bakry-Émrey criterion is not fulfilled. We consider twisted gradients, or equivalently, twisted metrics on the tangent space, by a section of $\mathrm{GL}(TM)$. This operation does not change the stochastic diffusion on M but creates a new translation on the tangent space, with associated generators and semi-groups. Under assumptions on these twists, which replace the Bakry-Émery condition, we can obtain intertwining relations at the level of semi-groups, between $\mathbf{P}$ and a the new semi-group $\mathbf{Q}^B$. This intertwining can be translated to Brascamp-Lieb type inequalities for the measure μ, extending the unreachable ones, satisfied under the strict convexity assumption of the potential. This method has been developed in [2] is the case of Euclidean space and shows it potential their. In our work, we extend the method to manifold and to a larger class of twistings.

Let us summarize the content of this chapter. In Section 4.2, we introduce twisting, the associated semi-group and its generator. In Section 4.3 we have a discussion on the stochastic representation of the semi-group on bounded 1-forms. It gives rise to a decomposition of the generator. The goal of Section 4.4 is to find conditions for this decomposition to be the sum of a symmetric positive second order generator and a zero order potential. In Sections 4.5 and 4.6, we obtain conditions to have intertwining relations for the L^2 semi-groups on 1-forms. Theorem 4.5.2 is a generalization of Theorem 2.2 in [2], in a manifold setting, with the same kind of assumptions: conditions of symmetry and positiveness of the second order operator and bound on the potential. Theorem 4.6.1 extends this result when the second order operator is not symmetric nor non-negative. We achieve to release all assumptions over the second order operator by a stronger bound on the potential. These intertwinings are applied in Theorems 4.5.3 and 4.6.3 to obtain generalized Brascamp-Lieb and Poincaré inequalities. We finish in Section 4.7 with several illustrations of measure which fail Bakry-Émery criterion in different ways and for which our method brings bounds on the spectral gap.

4.2 Twisted processes and semi-groups

Let B be a smooth section of $\mathrm{GL}(TM)$, i.e for all $x \in M$, $B(x)$ is an isomorphism of T_xM. The section B is used to twist the semi-group so as to obtain an intertwining relation even when the Bakry-Émery criterion is not satisfied. In this section, we are going to construct the tree levels, process, generator and semi-group, and prove a commutation at the level of generators. The intertwining relation at the level of semi-groups will be treated in Section 4.5 and 4.6. Firstly, we define the B-parallel translation above X^x by conjugation as:

$$/\!\!/_t^B = B(X_t^x)\, /\!\!/_t\, B(x)^{-1} : T_xM \to T_{X_t^x}M. \tag{4.2.1}$$

This new translation is also a diffusion on TM and we can calculate its generator on 1-forms, denoted by $L^{/\!/,B}$.

Proposition 4.2.1. *The generator on 1-forms of the B-parallel translation is given by*

$$L^{/\!/,B}\alpha = L^{/\!/}\alpha + 2(B^{-1})^*\nabla B^* \cdot \nabla\alpha + (B^{-1})^*(L^{/\!/}B^*)\alpha,\ \forall\alpha \in \Gamma(T^*M).$$

with the contraction $\nabla B^* \cdot \nabla\alpha = \sum_i \nabla_{e_i}B^* \cdot \nabla_{e_i}\alpha$ for any orthonormal basis $(e_i)_i$.

Proof. For all 1-form α, $w \in T_xM$ and $t \geq 0$, we have :

$$\begin{aligned}\langle\alpha, /\!/_t^B w\rangle &= \langle\alpha, B(X_t^x) /\!/_t B^{-1}(x)w\rangle \\ &= \langle B^*\alpha, B(x)^{-1}w\rangle\end{aligned}$$

Using the diffusion property of $/\!/_t$, we have :

$$d\langle\alpha, /\!/_t^B w\rangle \overset{(m)}{=} \langle L^{/\!/}(B^*\alpha), /\!/_t B^{-1}(x)w\rangle dt.$$

By definition of $L^{/\!/,B}$, we have :

$$d\langle\alpha, /\!/_t^B w\rangle \overset{(m)}{=} \langle L^{/\!/,B}\alpha, /\!/_t^B w\rangle dt.$$

Together, it yields :

$$L^{/\!/,B}\alpha = (B^*)^{-1}L^{/\!/}\left(B^*\alpha\right).$$

This ends the proof. □

Unlike the parallel translation, the B-parallel translation is not an isometry for the Riemmanian metric. Actually, it is not adapted to the Riemannian metric. To get back to a notion of isometric translation along curves, we need to twist the metric too and use the B-twisted metric: for all $v, w \in T_xM$

$$\langle v, w\rangle_B = \langle B^{-1}(x)v, B^{-1}(x)w\rangle. \tag{4.2.2}$$

However, the twisted-parallel translation $/\!/^B$ is still not the Levi-Civita parallel translation associated to the B-metric but we have a simple relation between them. Let us denote ∇^B the connexion associated to $/\!/^B$. It satisfies :

$$\nabla^B = \nabla - (\nabla B)B^{-1}. \tag{4.2.3}$$

Its torsion T^B is generically non-vanishing and satisfies : for all $X, Y \in \Gamma(TM)$,

$$T^B(X,Y) = (\nabla_Y B)B^{-1}X - (\nabla_X B)B^{-1}Y. \tag{4.2.4}$$

On the other hand, the Levi-Civita connexion $\nabla^{B\mathcal{LC}}$, satisfies : for all $X, Y, Z \in \Gamma(TM)$,

$$\left\langle \nabla_X^{B\mathcal{LC}} Y, Z \right\rangle_B = \left\langle \nabla_X^B Y - \frac{1}{2} T^B(X,Y), Z \right\rangle + \frac{1}{2} \left\langle T^B(X,Z), Y \right\rangle_B + \frac{1}{2} \left\langle T^B(y,Z), x \right\rangle_B . \tag{4.2.5}$$

Hence both connexions differ not only from a torsion term but also from a symmetric part. It is not a torsion skew symmetric case (see [32]). This also means that a twist is not only a change of metric.

Now, as in the non-twisted case, the next step is to define the B-deformed parallel translation as:

$$W_t^B = B(X_t^x) W_t(x) B^{-1} : T_x M \to T_{X_t^x} M. \tag{4.2.6}$$

This definition uses the previous definition of the deformed parallel translation, but, as a straightforward calculation shows, W_t^B could have been defined as in (3.2.4) by a stochastic covariant equation :

$$D_t^B W_t^B = -\mathcal{M}_B^* W_t^B \tag{4.2.7}$$

where D_t^B stands for the B-covariant derivative $/\!/_t^B d \left(/\!/_t^{B^{-1}} \right)$ and $\mathcal{M}_B^*$ is the adjoint of

$$\mathcal{M}_B = (B^*)^{-1} \mathcal{M} B^*. \tag{4.2.8}$$

Again, this translation is a diffusion in TM. As for $/\!/_t^B$, the same calculation shows that its generator on 1-forms, $L^{W,B}$, is conjugated to the generator L^W :

$$L^{W,B} = (B^*)^{-1} L^W B^*.$$

This gives a first decomposition of $L^{W,B}$.

Proposition 4.2.2. *The B-deformed parallel translation is a diffusion with generator on 1-forms*

$$L^{W,B} = L^{/\!/,B} - \mathcal{M}_B \tag{4.2.9}$$

The argument of conjugacy shows that $L^{W,B}$ and L are intertwined by $(B^*)^{-1}d$:

$$(B^*)^{-1} dL = L^{W,B} (B^*)^{-1} d. \tag{4.2.10}$$

The generator $L^{W,B}$ can be extended in a L^2-sense. We denote by $\langle \cdot, \cdot \rangle_B$ the intertwined-metric on 1-forms: for two 1-forms α, β,

$$\langle \alpha, \beta \rangle_B = \langle B^* \alpha, B^* \beta \rangle, \tag{4.2.11}$$

and by $L^2(B,\mu)$ the space of measurable 1-forms α such that

$$\int_M |\alpha|_B^2 \, d\mu < +\infty. \tag{4.2.12}$$

As L^W, $L^{W,B}$ is also essentially self-adjoint, on $L^2(B,\mu)$ and is associated to a L^2 semi-group of diffusion on 1-forms, $\mathbf{Q}_t^B$. Under suitable conditions, it would generates a semi-group on smooth compactly supported 1-forms, also denoted by $\mathbf{Q}_t^B$, with the stochastic representation

$$\langle \mathbf{Q}_t^B \alpha, v\rangle = \mathbb{E}\left[\langle \alpha, W_t^B v\rangle \mathbb{1}_{t<\tau_x}\right]. \tag{4.2.13}$$

It is the topic of next section.

4.3 Existence of the $\mathcal{C}^0$ semi-group

In this section, we look for criterion of existence of the $\mathcal{C}^0$-semi-group $\mathbf{Q}^B$. As for the non-twisted case, the existence of the L^2-semi-group is not a problem but additional assumptions come out when we want a stochastic representation of the semi-group on bounded 1-forms or when we want to prove an intertwining relation. It also appear that in the non-twisted case, a unique assumption, the Bakry-Émery criterion, solves both questions. In our twisting method, we hope that we can find a new criterion as substitution.

Firstly, let us remark that the Bakry-Émery criterion is still a a sufficient condition of existence and intertwining.

Proposition 4.3.1. *Under the Bakry-Émery criterion, $\mathbf{Q}^B$ is well-defined by the formula* (4.2.13) *and is intertwined to $\mathbf{P}$ by $(B^*)^{-1}d$, i.e. for all $f \in \mathcal{C}_c^\infty(M)$, for all $t \geq 0$,*

$$(B^*)^{-1} d\mathbf{P}_t f = \mathbf{Q}_t^B \left((B^*)^{-1} df\right).$$

Proof. As in Proposition 3.4.2, the Bakry-Émery criterion prove the existence of the stochastic representation (4.2.13). For all $f \in \mathcal{C}_c^\infty(M)$, we have:

$$\begin{aligned}
(B^*)^{-1})d\mathbf{P}_t f &= (B^*)^{-1}\mathbf{Q}_t df \\
&= (B^*)^{-1}\mathbb{E}\left[\langle df, W_t \cdot\rangle\right] \\
&= \mathbb{E}\left[\langle df, W_t B^{-1}(x)\cdot\rangle\right] \\
&= \mathbb{E}\left[\langle (B^*)^{-1} df, B(X_t^x) W_t B^{-1}(x)\cdot\rangle\right] \\
&= \mathbb{E}\left[\langle (B^*)^{-1} df, W_t^B \cdot\rangle\right].
\end{aligned}$$

□

As explained, this results is not very satisfying because the goal of twisting is to obtain bypass the Bakry-Émery criterion. Furthermore, the potential $\mathcal{M}^B$ appearing in (4.2.7) is conjugated to $\mathcal{M}$ so they have the same eigenvalues. Then $\mathcal{M}^B$ does not seem useful to improve inequalities such as in Section 3.4 even if we could obtained the intertwining relation without using the Bakry-Émery criterion. In order to find a more relevant potential, we get back to the definition of W_t^B. The twisted covariant equation (4.2.7) hide the role of twisting . Let us establish the non-twisted one.

Proposition 4.3.2. *The B-deformed parallel translation satisfies the stochastic covariant equation :*

$$D_t W_t^B = -\left(\mathcal{M}_B - L^{/\!/}(B)B^{-1}\right) W_t^B v \, dt + \left(\nabla_{d_m X_t^x} B\right)(B^{-1}) W_t^B v, \tag{4.3.1}$$

where $d_m X_t^x$ is the martingale part of the Ito derivative of X_t^x.

Proof. For all $x \in M$; $v \in T_x M$ and $t \geq 0$, we have :

$$\begin{aligned} D_t W_t^B v &= /\!/_t d\left(/\!/_t^{-1} W_t^B v\right) \\ &= /\!/_t d\left(/\!/_t^{-1} B(X_t^x) /\!/_t /\!/_t^{-1} W_t B(x)^{-1} v\right) \\ &= /\!/_t d\left(/\!/_t^{-1} B(X_t^x) /\!/_t\right) /\!/_t^{-1} W_t B(x)^{-1} v + B(X_t^x) D_t W_t B(x)^{-1} v + 0 \end{aligned}$$

In the first term, we recognize the stochastic covariant derivative in $\mathrm{End}(TM)$:

$$D_t B(X_t^x) = /\!/_t d\left(/\!/_t^{-1} B(X_t^x) /\!/_t\right) /\!/_t^{-1}.$$

The second term is given by (3.2.4). There is no quadratic term as W_t has finite variations. Then we have :

$$D_t W_t^B v = \left(L^{/\!/} B(X_t^x) dt + \nabla_{d_m X_t^x} B(X_t^x)\right) W_t B(x)^{-1} v - B(X_t^x) \mathcal{M} W_t B(x)^{-1} v dt.$$

□

This equation is less easy to handle because of the martingale terms which does not vanish. The potential involved in the finite variations part of $D_t W_t^B$ is a new one. We denote it by

$$M_B = \mathcal{M}_B - (B^*)^{-1} L^{/\!/}(B^*). \tag{4.3.2}$$

Thinking to the calculation of Section 3.4, this potential seems more relevant in a stochastic point of view : the growth of $|W_t^B|^2$ will be controlled by its first eigenvalue. In fact, we have something a bite more complicated because M_B is not necessarily symmetric and because the martingale part will take part. We denote

$$\rho_B = \inf_{x\in M} \left\{\text{smallest eigenvalue of } \left(B^* M_B (B^*)^{-1}\right)^s\right\}, \tag{4.3.3}$$

the analogous of ρ. We denote equally $\|\cdot\|_{HS}$ the Hilbert-Schmidt norm on $\mathrm{End}(TM)$ and the associated operator norm on $\mathcal{L}(TM, \mathrm{End}(TM))$. $\|\cdot\|_{HS,\infty}$ is its supremum over M.

Proposition 4.3.3. *Assume that $\rho_B - \|(\nabla.B^*)(B^*)^{-1}\|^2_{HS,\infty} > k > -\infty$, then $\mathbf{Q}^B$ is well-defined by* (4.2.13) *and for all $t \geq 0$ and α continuous bounded 1-form, we have :*

$$\|\mathbf{Q}^B_t \alpha\|_\infty \leq e^{-kt}\|\alpha\|_\infty.$$

Proof. For all $x \in M$ and $v \in T_xM$, we have :

$$\begin{aligned} d|W^B_t v|^2 &= 2\langle W^B_t v, DW^B_t v\rangle + \langle DW^B_\cdot v, DW^B_\cdot v\rangle_t \\ &= -2\langle W^B_t v, M_B W^B_t v\rangle dt + (m) + 2\sum_i |(\nabla_{e_i} B^*(X_t))(B^*)^{-1}(X_t) W^B_t v|^2 dt \\ &= -2\langle W^B_t v, M_B W^B_t v\rangle dt + (m) + 2\|(\nabla.B^*(X_t))(B^*)^{-1}(X_t) W^B_t v\|^2_{HS} dt \end{aligned}$$

with $(e_i)_i$ an orthonormal basis. Let τ_n be a localizing sequence for the martingale part. We have :

$$\mathbb{E}\left[|W^B_t v|^2 \mathbb{1}_{t\leq\tau_n}\right] \leq |v|^2 - 2k\int_0^t \mathbb{E}\left[|W^B_s v|^2 \mathbb{1}_{s\leq\tau_n}\right] ds. \tag{4.3.4}$$

By Grönwall lemma and monotonous convergence, we have :

$$\mathbb{E}\left[|W^B_t v|^2 \mathbb{1}_{t\leq\tau}\right] \leq e^{-2kt}|v|^2. \tag{4.3.5}$$

This is sufficient to defined the semi-group as (4.2.13) and obtain the upper bound. □

This condition is sufficient for the existence of the semi-group $\mathbf{Q}^B$ but is not sufficient to prove the intertwining. That is why we will focus on the L^2-semi-group in the next sections. The main interest of this

The new multiplicative potential M_B leads to a second decomposition of the generator $L^{W,B}$:

$$L^{W,B} = L^{/\!/}_B - M_B. \tag{4.3.6}$$

where $L^{/\!/}_B$ is defined by :

$$L^{/\!/}_B = L^{/\!/} + 2(B^*)^{-1}\nabla B^* \cdot \nabla \cdot \tag{4.3.7}$$

From the operators point of view, this new split seems more satisfying too. Indeed, the potential M_B contains all the zero-order terms of $L^{W,B}$ and only them. Of course, this potential is also more relevant in an heuristic way, if we think about the Euclidean study [2]. It is also linked to the work on Lyapunov functions in [10] and [24] as we will see in Section 4.4. So, this potential is the natural candidate for a generalization of Bakry-Émery criterion.

4.4 Symmetry and positiveness of $-L_B^{/\!/}$

First, as we noticed, $L^{W,B}$ is conjugated to L^W, and so, is self-adjoint in $L^2(B,\mu)$. For the same reason, in the subspace of twisted differential-forms

$$\{(B^*)^{-1}df; f \in \mathcal{C}_c^\infty(M)\},$$

we additionally prove the non-positiveness of $L^{W,B}$:

$$\begin{aligned}\int_M \langle (B^*)^{-1}df, L^{W,B}(B^*)^{-1}df\rangle_B \, d\mu &= \int_M \langle df, L^W df\rangle \, d\mu \\ &= \int_M \langle df, d(Lf)\rangle \, d\mu \\ &= -\int_M (Lf)^2 \, d\mu.\end{aligned}$$

The classical result, in non-twisted cases, use the decomposition of L^W as the sum of a symmetric non-positive operator and a potential bounded from below. So we are looking for conditions such that $L_B^{/\!/}$ is symmetric with respect to the B-twisted metric. This is not trivial, even in the subspace of twisted gradients. First, by integration by parts for the horizontal Laplacian, we have

$$\int_M \langle (-L^{/\!/})\alpha, \beta\rangle \, d\mu = \int_M \langle \nabla\alpha, \nabla\beta\rangle \, d\mu \tag{4.4.1}$$

with $\langle \nabla\alpha, \nabla\beta\rangle = \sum_i \langle \nabla_{e_i}\alpha, \nabla_{e_i}\beta\rangle$, with $(e_i)_i$ any orthonormal basis. Then, on one hand, we have :

$$\begin{aligned}\int_M \langle (-L^{/\!/})\alpha, \beta\rangle_B \, d\mu &= \int_M \langle (-L^{/\!/})\alpha, (B^*)^t B^*\beta\rangle \, d\mu \\ &= \int_M \langle \nabla\alpha, \nabla((B^*)^t B^*\beta)\rangle \, d\mu \\ &= \int_M \langle \nabla\alpha, (B^*)^t B^*\nabla\beta\rangle \, d\mu + \int_M \langle \nabla\alpha, \nabla((B^*)^t B^*)\beta\rangle \, d\mu \\ &= \int_M \langle \nabla\alpha, \nabla\beta\rangle_B \, d\mu + \int_M \langle \nabla\alpha, \nabla(B^*)^t B^*\beta\rangle \, d\mu\end{aligned}$$

$$+\int_M \langle \nabla\alpha, (B^*)^t \nabla(B^*)\beta\rangle \, d\mu$$

where $(B^*)^t$ denotes the dual map of B^* with respect to scalar products on T^*M. On the other hand, we have :

$$\begin{aligned} -\int_M \left\langle 2(B^{-1})^* \nabla B^* \cdot \nabla\alpha, \beta \right\rangle_B \, d\mu &= -2\int_M \langle \nabla B^* \cdot \nabla\alpha, B^*\beta\rangle \, d\mu \\ &= -2\int_M \left\langle \nabla\alpha, (\nabla B^*)^t \, B^*\beta \right\rangle d\mu. \end{aligned}$$

This yields

$$\int_M \left\langle \left(-L_B^{/\!/}\right)\alpha, \beta \right\rangle_B \, d\mu = \int_M \langle \nabla\alpha, \nabla\beta\rangle_B \, d\mu - \int_M \langle B^*\nabla\alpha, \mathcal{B}(B^*\beta)\rangle \, d\mu. \quad (4.4.2)$$

where

$$\mathcal{B} = \left((\nabla B^*)(B^*)^{-1}\right)^t - (\nabla B^*)(B^*)^{-1} \quad (4.4.3)$$

We immediately get this first criterion of symmetry and non-negativeness.

Proposition 4.4.1. *If $\mathcal{B} = 0$, then the generator $-L_B^{/\!/}$ is symmetric with respect to $\langle\cdot,\cdot\rangle_B$, non-negative and we have:*

$$-\int_M \langle L_B^{/\!/}\alpha, \beta\rangle_B \, d\mu = \int_M \langle \nabla\alpha, \nabla\beta\rangle_B \, d\mu \quad (4.4.4)$$

The criterion of Proposition 4.4.1 is obviously not necessary but it gives a condition easy to check and not to constraining as we will see.

On other way to find a condition of symmetry is to look at the potential rather than the operator. The operator $L^{W,B}$ and the potential $\mathcal{M}_B$ are symmetric with respect to the B-metric and we have :

$$L_B^{/\!/} = L^{W,B} + \mathcal{M}_B - (B^*)^{-1} L^{/\!/}(B^*). \quad (4.4.5)$$

So a necessary and sufficient condition for the B-symmetry of $L_B^{/\!/}$ is the B-symmetry of the potential $(B^*)^{-1} L^{/\!/}(B^*)$. But unlike the condition of Proposition 4.4.1, this is not a sufficient condition for positiveness. For example, one can look at $(\mathbb{R}_+^*)^2$ with the potential $V(x,y) = x + y$ and the twist

$$B^* = \begin{pmatrix} \varphi & \varphi \\ 1 & e^V \end{pmatrix},$$

where φ is positive such that $L\varphi \neq 0$. The associated $L_B^{/\!/}$ is symmetric but is not non-negative.

The following result is immediate and gives examples satisfying the condition of Proposition 4.4.1.

Proposition 4.4.2. *If $B(x) = b(x)\,\mathrm{id}_{T_xM}$ for some $b \in \mathcal{C}^\infty(M)$, then $\mathcal{B} = 0$.*

Proof. For all $x \in M$, we have $(\nabla B^*)(B^*)^{-1} = b^{-1}\nabla b \otimes \mathrm{id}_{T_x^*M}$. It is clearly symmetric. $\square$

In the same spirit, we have a result for product manifolds endowed with a product-compatible metric.

Proposition 4.4.3. *Let $(M_i, g_i)_{1\le i\le n}$ be Riemmanian manifolds and (M, g) the product manifold endowed with the product metric. If $B(x) = \sum_{i=1}^n b_i(x)\,\mathrm{id}_{T_{x_i}M_i}$ for some functions $b_i \in \mathcal{C}^\infty(M)$, then $\mathcal{B} = 0$.*

This results allows us to consider non-homothetic diagonal twists in $\mathbb{R}^n$. Keep in mind that for non-product manifolds, this result may be invalid. For example, in the Heisenberg group endowed with its canonical left-invariant metric (see Section 5.4), a straightforward calculation proves that the only diagonal twists satisfying $\mathcal{B} = 0$ are homothetic. It seems difficult to find other kinds of examples of twist satisfying the criterion $\mathcal{B} = 0$. Nevertheless, this class of twists is directly linked to the study of Lyapunov functions in [24]. Actually, for a twist $B(x) = b(x)\,\mathrm{id}_{T_xM}$, the eigenvalues of M_B are the eigenvalue of $\mathcal{M}$ shifted by $-b^{-1}L(b)$, which appears in their calculation. We will see in the last section that this class of twists allows us to treat various types of examples.

Finally, remark that the tensor $\mathcal{B} \in T^*M \otimes TM \otimes T^*M$ can be related to the torsion T^B, or more easily to the difference between ∇ and ∇^B. Indeed, we have : for all $X, Y \in \Gamma(TM)$,

$$\mathcal{B}^\#(X, Y^\flat) = B^{-1}\left(\nabla_X - \nabla_X^B\right) BY - B^{-1}\left(\nabla_X - \nabla_X^B\right)^t BY. \tag{4.4.6}$$

Thought, this formula does not really clarified this link.

4.5 Intertwining: a symmetric positive case

The goal of this section, is to prove the intertwining relation and Poincaré inequality under the assumption $\mathcal{B} = 0$. According to Proposition 4.4.1, $-L_B^{/\!/}$ is symmetric, non-negative, with respect to $\langle\cdot,\cdot\rangle_B$. As $L^{W,B}$ is symmetric with respect to this metric, then M_B is symmetric too. We still denote by ρ_B the infimum over M of the smallest eigenvalue of $B^*M_B(B^*)^{-1}$:

$$\rho_B = \inf_{x\in M}\left\{\text{smallest eigenvalue of } B^*M_B(B^*)^{-1}\right\}. \tag{4.5.1}$$

We also assume that ρ_B is bounded from below. As we already said, the generator $L^{W,B}$ is essentially self-adjoint. With this new assumption, $L^{W,B}$ is the sum of a symmetric non-negative operator $L_B^{/\!/}$ and a bounded from below potential M_B. So we could obtain a new proof of the the essential self-adjointness as a generalization of proof of Strichartz in [75]. In order to obtain the intertwining relation, we need to show that $(B^*)^{-1}d\mathbf{P}_t f$ is the unique L^2 strong solution to the Cauchy problem

$$\begin{cases} \partial_t F = L^{W,B} F \\ F(\cdot,0) = G \in L^2(B,\mu) \end{cases}$$

where the mapping $t \mapsto F(\cdot,t)$ is continuous from $\mathbb{R}_+$ to $L^2(B,\mu)$. Remark that we are looking for a strong solution : in this Cauchy problem, $L^{W,B}$ has to be understood as a differential operator and not an L^2 operator. Actually, as we do not know the domain of $L^{W,B}$, we cannot use the uniqueness in the sense of self-adjoint operator.

Proposition 4.5.1. *Assume that $\mathcal{B} = 0$ and that M_B is uniformly bounded from below. Let F be a solution of the L^2 Cauchy problem above. Then, we have*

$$F(\cdot,t) = \mathbf{Q}_t^B(G), t \geq 0.$$

Proof. We generalize the argument of [57] and [2] which deal respectively with the case of a Laplacian in a Riemannian manifold and the case of our operator $L^{W,B}$ in $\mathbb{R}^n$. By linearity, it is sufficient to show the uniqueness of the solution for the zero initial condition. Replacing the solution F by $e^{-\rho_B t}F$, let us assume that M_B is positive semi-definite. For every $\phi \in \mathcal{C}_c^\infty(M)$ and $\tau > 0$, we have:

$$\begin{aligned} \int_0^\tau \int_M \phi^2 \langle F, L_B^{/\!/} F\rangle_B \, d\mu \, dt &= \int_0^\tau \int_M \phi^2 \langle F, (L^{W,B} + M_B)F\rangle_B \, d\mu \, dt \\ &= \int_0^\tau \int_M \phi^2 \frac{1}{2} \partial_t |F|_B^2 \, d\mu \, dt + \int_0^\tau \int_M \phi^2 \langle F, M_B F\rangle_B \, d\mu \, dt \\ &\geq \int_M \phi^2 \frac{1}{2} |F(\cdot,\tau)|_B^2 \, d\mu. \end{aligned}$$

On the other hand, by the integration by parts formula of Proposition 4.4.1, we have

$$\begin{aligned} \int_0^\tau \int_M \phi^2 \langle F, L_B^{/\!/} F\rangle_B \, d\mu \, dt &= -\int_0^\tau \int_M \langle \nabla(\phi^2 F), \nabla F\rangle_B \, d\mu \, dt \\ &= -\int_0^\tau \int_M \phi^2 |\nabla F|_B^2 \, d\mu \, dt \\ &\quad - 2\int_0^\tau \int_M \langle \nabla\phi \otimes F, \phi \nabla F\rangle_B \, d\mu \, dt. \end{aligned}$$

By Cauchy-Schwarz inequality, we have for every $\lambda > 0$,

$$2\,|\langle \nabla\phi \otimes F, \phi\nabla F\rangle_B| \leq \lambda |\nabla\phi|_2^2 |F|_B^2 + \frac{1}{\lambda}\phi^2 |\nabla F|_B^2. \tag{4.5.2}$$

Combining the above inequalities, in the particular case of $\lambda = 2$, we obtain

$$\frac{1}{2}\int_M \phi^2 |F(\cdot,\tau)|_B^2 \, d\mu \leq -\frac{1}{2}\int_0^\tau \int_M \phi^2 |\nabla F|_B^2 \, d\mu \, dt + 2\int_0^\tau \int_M |\nabla \phi|_2^2 |F|_B^2 \, d\mu \, dt.$$

By completeness of M, there exists a sequence of cut-off functions $(\phi_n)_{n\in\mathbb{N}}$ in $\mathcal{C}_c^\infty(M)$ such that $(\phi_n)_n$ converge to 1 pointwise and $\|\nabla \phi_n\|_\infty \to 0$ as $n \to \infty$. Plugging this sequence in the previous inequality, gives

$$\int_M |F(\cdot,\tau)|_B^2 \, d\mu = 0, \tau > 0. \tag{4.5.3}$$

Hence $F = 0$ in $\mathcal{C}^0\left(\mathbb{R}_+, L^2(B,\mu)\right)$. □

Theorem 4.5.2. *Assume that $\mathcal{B} = 0$ and that M_B is uniformly bounded from below. Then the semi-groups $\mathbf{P}$ and $\mathbf{Q}^B$ are intertwined by $(B^*)^{-1}d$: for all $t \geq 0$ and $f \in \mathcal{C}_c^\infty(M)$*

$$(B^*)^{-1} d\mathbf{P}_t f = \mathbf{Q}_t^B \left((B^*)^{-1} df\right).$$

Proof. The main argument is to prove that $F(\cdot,t) = (B^*)^{-1}d\mathbf{P}_t f$ is a solution of the previous L^2 Cauchy problem with initial condition $G = (B^*)^{-1}df$. First, G is in $L^2(B,\mu)$ since f is compactly supported. For all $t > 0$, we have:

$$\begin{aligned}\int_M |F(\cdot,t)|_B^2 \, d\mu &= \int_M |(B^*)^{-1} d\mathbf{P}_t f|_B^2 \, d\mu \\ &= \int_M |d\mathbf{P}_t f|^2 \, d\mu \\ &= -\int_M \mathbf{P}_t f L \mathbf{P}_t f \, d\mu,\end{aligned}$$

which is finite since $\mathbf{P}_t f \in \mathcal{D}(L) \subset L^2(\mu)$. So $F(\cdot,t)$ is in $L^2(B,\mu)$ for every $t > 0$. Besides, the L^2 continuity is proven by the same calculation, since for every $t, s \geq 0$,

$$\int_M |F(\cdot,t) - F(\cdot,s)|_B^2 \, d\mu = -\int_M (\mathbf{P}_t f - \mathbf{P}_s f) L(\mathbf{P}_t f - \mathbf{P}_s f) \, d\mu. \tag{4.5.4}$$

By spectral theorem, this is upper bounded by $(\sup_{x\in\mathbb{R}_+} |\sqrt{x}(e^{-tx} - e^{-sx})|)^2 \|f\|_2^2$ which tends to zero as s tends to $t > 0$ (see [70] for more details on spectral theorem). For the right-continuity in $t = 0$, we use that

$$\int_M (\mathbf{P}_s f - f) L(\mathbf{P}_s f - f) \, d\mu = \int_0^s \int_0^s \int_M \mathbf{P}_t L f \mathbf{P}_u L^2 f \, d\mu \, du \, dt. \tag{4.5.5}$$

Finally, the commutation property (4.2.10), yields

$$\partial_t F = (B^*)^{-1} dL\mathbf{P}_t f = L^{W,B}\left((B^*)^{-1} d\mathbf{P}_t f\right) = L^{W,B} F.$$

The result follow by the uniqueness of the solution of the Cauchy problem. □

With this intertwining relation, we are now able to prove some functional inequalities. We get back to the assumptions of ergodicity and finite measure from Proposition 3.3.1. We have an integral representation of the covariance, using the semi-group $\mathbf{Q}^B$ instead of $\mathbf{Q}$: for all $f, g \in \mathcal{C}_c^\infty(M)$,

$$\mathrm{Cov}_\mu(f,g) = \int_0^{+\infty} \left(\int_M \left\langle (B^*)^{-1} df, \mathbf{Q}_t^B((B^*)^{-1} dg) \right\rangle_B d\mu \right) dt. \tag{4.5.6}$$

The main application of this covariance's representation is a generalization of an inequality due to Brascamp and Lieb, in [20], known as Brascamp-Lieb inequality.

Theorem 4.5.3 (Generalized Brascamp-Lieb inequality - symmetric case)**.** *Assume that $\mathcal{B} = 0$ and that M_B is positive definite, then for every $f \in \mathcal{C}_0^\infty(M)$, we have :*

$$\mathrm{Var}_\mu(f) \leq \int_M \langle df, \left((B^* M_B (B^*)^{-1}\right)^{-1} df\rangle \, d\mu.$$

Firstly, we need a little lemma.

Lemma 4.5.4. *Let C and D be symmetric non-negative operators such that D and $C + D$ are invertible. Then we have*

$$0 \leq D^{-1} - (C+D)^{-1}.$$

Proof. We have:

$$D^{-1} - (C+D)^{-1} = (C+D)^{-1} C D^{-1},$$

and we have

$$\langle (C+D)^{-1} C D^{-1} \alpha, \alpha \rangle = \langle C D^{-1} \alpha, (C+D)^{-1} \alpha \rangle.$$

Letting $(C+D)^{-1}\alpha = \beta$ this rewrites as

$$\begin{aligned} \langle C D^{-1}(C+D)\beta, \beta\rangle &= \langle C D^{-1} C\beta, \beta\rangle + \langle C\beta, \beta\rangle \\ &= \langle D^{-1} C\beta, C\beta\rangle + \langle C\beta, \beta\rangle \geq 0, \end{aligned}$$

since D^{-1} and C are non-negative. □

Proof of Theorem 4.5.3. First, let assume that ρ_B is positive. This implies that for all 1-form α, we have

$$\int_M \langle (-L^{W,B})\alpha, \alpha \rangle_B \, d\mu \geq \rho_B \int_M |\alpha|_B^2 \, d\mu. \tag{4.5.7}$$

So $-L^{W,B}$ is essentially self-adjoint and bounded from below by $\rho_B \, \mathrm{id}$. Then it is invertible in $L^2(B,\mu)$ i.e given any smooth compactly supported 1-form α, the Poisson equation $-L^{W,B}\beta = \alpha$ admits a unique solution β in the domain of $L^{W,B}$ which has the following integral representation:

$$\beta = \int_0^{+\infty} \mathbf{Q}_t^B(\alpha) \, dt = (-L^{W,B})^{-1}\alpha. \tag{4.5.8}$$

Using the variance representation formula (4.5.6), we have

$$\begin{aligned}
\mathrm{Var}_\mu(f) &= \int_0^\infty \left(\int_M \left\langle (B^*)^{-1}df, \mathbf{Q}_t^B \left((B^*)^{-1}df \right) \right\rangle_B \, d\mu \right) dt \\
&= \int_M \left\langle (B^*)^{-1}df, \int_0^\infty \mathbf{Q}_t^B \left((B^*)^{-1}df \right) \, dt \right\rangle_B \, d\mu \\
&= \int_M \left\langle (B^*)^{-1}df, (-L^{W,B})^{-1} \left((B^*)^{-1}df \right) \right\rangle_B \, d\mu \\
&= \int_M \left\langle (B^*)^{-1}df, (-L_B^{/\!/} + M_B)^{-1} \left((B^*)^{-1}df \right) \right\rangle_B \, d\mu
\end{aligned}$$

Using Lemma 4.5.4 to $C = -L_B^{/\!/}$ and $D = M_B$, we have:

$$\begin{aligned}
\mathrm{Var}_\mu(f) &\leq \int_M \left\langle (B^*)^{-1}df, M_B^{-1} \left((B^*)^{-1}df \right) \right\rangle_B \, d\mu \\
&\leq \int_M \left\langle df, \left(B^* M_B (B^*)^{-1} \right)^{-1} df \right\rangle \, d\mu
\end{aligned}$$

Now, when the operator M_B is not uniformly bounded from below by a positive constant, we need to regularize. For all $\varepsilon > 0$, the operator $\varepsilon \, \mathrm{id} - L^{W,B}$ is invertible and we have the following integral representation for all 1-form α:

$$(\varepsilon \, \mathrm{id} - L^{W,B})^{-1}\alpha = \int_0^{+\infty} e^{-\varepsilon t} \mathbf{Q}_t^B \alpha \, dt. \tag{4.5.9}$$

Similarly, $(\varepsilon \, \mathrm{id} - L)$ is invertible on the sub-space of centred functions and we have the integral representation for all centred $f \in \mathcal{C}_c^\infty(M)$:

$$(\varepsilon \, \mathrm{id} - L)^{-1} f = \int_0^{+\infty} e^{-\varepsilon t} \mathbf{P}_t f \, dt := g_\varepsilon. \tag{4.5.10}$$

We have:

$$
\begin{aligned}
\mathrm{Var}_\mu(f) &= \int_M f^2\, d\mu \\
&= \int_M f(\varepsilon \,\mathrm{id} - L) g_\varepsilon \, d\mu \\
&= \varepsilon \int_M f g_\varepsilon \, d\mu + \int_M f(-L) \left(\int_0^{+\infty} e^{-\varepsilon t} \mathbf{P}_t f \, dt \right) d\mu \\
&= \varepsilon \int_M f g_\varepsilon \, d\mu + \int_0^{+\infty} e^{-\varepsilon t} \int_M f(-L) \mathbf{P}_t f \, d\mu \, dt \\
&= \varepsilon \int_M f g_\varepsilon \, d\mu + \int_0^{+\infty} e^{-\varepsilon t} \int_M \left\langle (B^*)^{-1} df, \mathbf{Q}_t^B((B^*)^{-1} df) \right\rangle_B d\mu \, dt \\
&= \varepsilon \int_M f g_\varepsilon \, d\mu + \int_M \left\langle (B^*)^{-1} df, \int_0^{+\infty} e^{-\varepsilon t} \mathbf{Q}_t^B((B^*)^{-1} df) \, dt \right\rangle_B d\mu \\
&= \varepsilon \int_M f g_\varepsilon \, d\mu + \int_M \left\langle (B^*)^{-1} df, (\varepsilon \,\mathrm{id} - L^{W,B})^{-1}((B^*)^{-1} df) \right\rangle_B d\mu.
\end{aligned}
$$

We can apply Lemma 4.5.4 to $\varepsilon \,\mathrm{id} - L^{W,B} = \varepsilon \,\mathrm{id} - L_B^{/\!/} + M_B$. We have:

$$
\mathrm{Var}_\mu(f) \leq \varepsilon \|f\|_{L^2(\mu)} \|g_\varepsilon\|_{L^2(\mu)} + \int_M \left\langle (B^*)^{-1} df, (M_B)^{-1}((B^*)^{-1} df) \right\rangle_B d\mu.
$$

Finally, we have

$$
\varepsilon \|g_\varepsilon\|_{L^2(\mu)} = \left\| \int_0^{+\infty} e^{-t} \mathbf{P}_{t/\varepsilon} f \, dt \right\|_{L^2(\mu)} \leq \int_0^{+\infty} \int_M e^{-t} (\mathbf{P}_{t/\varepsilon} f)^2 \, d\mu \, dt.
$$

By ergodicity of $\mathbf{P}$ and dominated convergence, this term converges to 0 as $\varepsilon \to 0$. This ends the proof. □

An immediate corollary of this theorem is the Poincaré inequality.

Theorem 4.5.5 (Poincaré inequality - symmetric case)**.** *Assuming that $\mathcal{B} = 0$ and that ρ_B is positive, for all $f \in \mathcal{C}_c^\infty(M)$, we have*

$$
\mathrm{Var}_\mu(f) \leq \frac{1}{\rho_B} \int_M |df|^2 \, d\mu,
$$

In the case where M_B is only positive and not uniformly bounded from below (i.e $\rho_B = 0$), this inequality is trivially true. Let us give an alternative proof which does not use the generalized Brascamp-Lieb inequality, and thus, avoids the inversion of $L^{W,B}$ and its integral representation.

Proof. Using a time change and the symmetry of the semi-group $\mathbf{Q}^B$, we have a new representation of the variance:

$$\begin{aligned}\mathrm{Var}_\mu(f) &= 2\int_0^\infty \left(\int_M \left\langle (B^*)^{-1}df, \mathbf{Q}^B_{2t}((B^*)^{-1}df)\right\rangle_B \, d\mu\right) dt \\ &= 2\int_0^\infty \left(\int_M \left|\mathbf{Q}^B_t((B^*)^{-1}df)\right|_B^2 \, d\mu\right) dt.\end{aligned}$$

Let

$$\phi(t) = \int_M \left|\mathbf{Q}^B_t((B^*)^{-1}df)\right|_B^2 \, d\mu. \tag{4.5.11}$$

We have

$$\begin{aligned}\phi'(t) &= 2\int_M \left\langle \mathbf{Q}^B_t((B^*)^{-1}df), L^{W,B}\mathbf{Q}^B_t((B^*)^{-1}df)\right\rangle_B \, d\mu \\ &= 2\int_M \left\langle \mathbf{Q}^B_t((B^*)^{-1}df), (L^{/\!/}_B)\mathbf{Q}^B_t((B^*)^{-1}df)\right\rangle_B \, d\mu \\ &\quad - 2\int_M \left\langle \mathbf{Q}^B_t((B^*)^{-1}df), M_B\mathbf{Q}^B_t((B^*)^{-1}df)\right\rangle_B \, d\mu \\ &= -2\int_M \left|\nabla\mathbf{Q}^B_t((B^*)^{-1}df)\right|_B^2 \, d\mu \\ &\quad - 2\int_M \left\langle \mathbf{Q}^B_t((B^*)^{-1}df), M_B\mathbf{Q}^B_t((B^*)^{-1}df)\right\rangle_B \, d\mu \\ &\leq -2\int_M \left\langle \mathbf{Q}^B_t((B^*)^{-1}df), M_B\mathbf{Q}^B_t((B^*)^{-1}df)\right\rangle_B \, d\mu \\ &\leq -2\rho_B\phi(t)\end{aligned}$$

By Grönwall lemma, this implies

$$\phi(t) \leq e^{-2\rho_B t}\phi(0) = e^{-2\rho_B t}\int_M |df|^2 \, d\mu. \tag{4.5.12}$$

Integrating on $\mathbb{R}_+$ ends the proof. □

We finish with an interpretation of the Poincaré inequality in terms of spectral gap.

Proposition 4.5.6. *Assume that $\mathcal{B} = 0$ and that ρ_B is positive then the spectral gap satisfies*

$$\lambda_1(-L, \mu) \geq \rho_B \tag{4.5.13}$$

This is a generalization to Riemannian manifolds of the Chen and Wang formula established in the one dimensional case in [27]. This spectral gap gives an exponential rate of convergence to equilibrium to the semi-group $\mathbf{P}$ in norm L^2 :

$$\|P_t f - \mu(f)\|_{L^2(\mu)} \leq e^{-\rho_B t}\|f - \mu(f)\|_{L^2(\mu)}, \tag{4.5.14}$$

much stronger than our ergodicity assumption.

4.6 Intertwining: general case

The goal of this section is to extend the results of Section 4.5 without the strong condition of Proposition 4.4.1. These results are more theoretical because we will not apply it to any example but they show that our twisting method is strong to perturbations and could be applied to a more general class of twist than the class of Proposition 4.4.2. Actually, we can release all assumptions on the second order operator if we are ready to strengthen the conditions on the potential M_B. In this case, the eigenvalue ρ_B is not a good criterion anymore. We need to find a quantity which offsets the lack of symmetry. For all 1-form α, according to (4.4.2), we have:

$$\begin{aligned}\int_M \langle L_B^{/\!/}\alpha, \alpha\rangle_B \, d\mu &= -\int_M |B^*\nabla\alpha|^2 \, d\mu + \int_M \langle B^*\nabla\alpha, \mathcal{B}(B^*\alpha)\rangle \, d\mu \\ &= -\int_M \left|B^*\nabla\alpha - \frac{1}{2}\mathcal{B}B^*\alpha\right|^2 \, d\mu + \int_M \frac{1}{4}\left|\mathcal{B}B^*\alpha\right|^2 \, d\mu \\ &\leq \int_M \langle B^*\alpha, N_B B^*\alpha\rangle \, d\mu\end{aligned}$$

with $\mathcal{B}$ defined in (4.4.3) and N_B the section of $\mathrm{End}(T^*M)$ defined by

$$N_B(x) = \frac{1}{4}\mathcal{B}(x)^t \cdot \mathcal{B}(x) \in \mathrm{End}(T_x^*M). \tag{4.6.1}$$

Hence, we have the following lower bound:

$$\int_M \left\langle \left(-L^{W,B}\right)\alpha, \alpha\right\rangle_B \, d\mu \geq \int_M \left\langle B^*\alpha, \left[\left(B^*M_B(B^*)^{-1}\right)\right)^s - N_B\right] B^*\alpha\right\rangle \, d\mu. \tag{4.6.2}$$

where $(B^*M_B(B^*)^{-1}))^s$ is the symmetric part of $B^*M_B(B^*)^{-1}$ with respect to the Riemannian metric. So the quantity we need to control seems to be the following:

$$\tilde{\rho}_B = \inf_{x\in M}\left\{\text{smallest eigenvalue of } \left(B^*M_B(B^*)^{-1}\right)\right)^s - N_B\right\}. \tag{4.6.3}$$

First, as in the symmetric case, we show the intertwining relation.

Theorem 4.6.1. *Assume that $(B^*M_B(B^*)^{-1}))^s - (1+\varepsilon)N_B$ is bounded from below for some $\varepsilon > 0$. Then the semi-groups $\mathbf{P}$ and $\mathbf{Q}^B$ are intertwined by $(B^*)^{-1}d$, i.e for every $f \in \mathcal{C}_c^\infty(M)$ and $t \geq 0$ we have :*

$$(B^{-1})^* d\mathbf{P}_t f = \mathbf{Q}_t^B\left((B^{-1})^* df\right).$$

Proof. The core of the proof is still the uniqueness of the solution of the same L^2 Cauchy problem. We assume that $(B^*M_B(B^*)^{-1}))^s - (1+\varepsilon)N_B$ is non-negative without any loss of generality. Let F be a solution with the zero initial condition. For $\phi \in \mathcal{C}_c^\infty$ and $\tau > 0$, as in the proof of Proposition 4.5.1, we have

$$\int_0^\tau \int_M \phi^2 \langle F, (L_B^{/\!/} - (1+\varepsilon)(B^*)^{-1} N_B B^*) F\rangle_B \, d\mu \, dt \geq \int_M \phi^2 \frac{1}{2} |F(\cdot,\tau)|_B^2 \, d\mu. \quad (4.6.4)$$

On the other hand, according to the formula (4.4.2), we have

$$\begin{aligned}
\int_M \phi^2 \langle F, L_B^{/\!/} F\rangle_B \, d\mu &= -\int_M \langle \nabla(\phi^2 F), \nabla F\rangle_B \, d\mu + \int_M \langle B^* \nabla F, \mathcal{B}(B^* \phi^2 F)\rangle \, d\mu \\
&= -\int_M \phi^2 |\nabla F|_B^2 \, d\mu + \int_M \phi^2 \langle B^* \nabla F, \mathcal{B}(B^* \phi^2 F)\rangle \, d\mu \\
&\quad - 2\int_M \langle \nabla\phi \otimes F, \phi \nabla F\rangle_B \, d\mu \\
&= -\int_M \phi^2 |B^* \nabla F - \frac{1}{2} \mathcal{B} B^* F|^2 \, d\mu + \int_M \phi^2 \langle F, N_B F\rangle \, d\mu \\
&\quad - 2\int_M \langle \nabla\phi \otimes F, \phi(\nabla F - \frac{1}{2}(B^*)^{-1} \mathcal{B} B^* F)\rangle_B \, d\mu \\
&\quad - 2\int_M \langle \nabla\phi \otimes F, \phi \frac{1}{2}(B^*)^{-1} \mathcal{B} B^* F\rangle_B \, d\mu.
\end{aligned}$$

According to Cauchy-Schwarz inequality, for every $\lambda, k > 0$, we have:

$$2|\langle \nabla\phi \otimes F, \phi(\nabla F - \frac{1}{2}(B^*)^{-1} \mathcal{B} B^* F)\rangle_B| \leq \lambda |\nabla\phi \otimes F|_B^2 + \frac{1}{\lambda} \phi^2 |B^* \nabla F - \frac{1}{2} \mathcal{B} B^* F|^2$$
$$2|\langle \nabla\phi \otimes F, \phi \frac{1}{2}(B^*)^{-1} \mathcal{B} B^* F\rangle_B| \leq k |\nabla\phi \otimes F|_B^2 + \frac{1}{k} \phi^2 |\frac{1}{2} \mathcal{B} B^* F|^2$$

So, we have:

$$\begin{aligned}
\int_M \phi^2 \langle F, L_B^{/\!/} F\rangle_B \, d\mu &\leq \left(\frac{1}{\lambda} - 1\right) \int_M \phi^2 |B^* \nabla F - \frac{1}{2} \mathcal{B} B^* F|^2 \, d\mu \\
&\quad + \left(1 + \frac{1}{k}\right) \int_M \phi^2 \langle F, N_B F\rangle \, d\mu + (\lambda + k) \int_M |\nabla\phi|^2 |F|_B^2 \, d\mu.
\end{aligned}$$

Combining the above inequalities, we obtain that there exists a $c > 0$ such that for every $\phi \in \mathcal{C}_c^\infty(M)$, and every $\tau > 0$

$$\frac{1}{2} \int_M \phi^2 |F(\cdot,\tau)|_B^2 \, d\mu \leq c \int_0^\tau \int_M |\nabla\phi|^2 |F|_B^2 \, d\mu \, dt. \quad (4.6.5)$$

Using a sequence of cut-off functions, we prove that $F = 0$. The end of the proof follows the proof of Theorem 4.5.2 without any differences. $\square$

Remark that under the condition of Proposition 4.6.1, $\tilde{\rho}_B$ is bounded from below, since N_B is non-negative. But unlike in Theorem 4.5.1, this proof requires a stronger condition.

Back to our assumptions of ergodicity and probability measure, the intertwining relation of Proposition 4.6.1 implies the covariance's representation (4.5.6). This brings Brascamp-Lieb and Poincaré type inequalities.

Theorem 4.6.2 (Poincaré inequality - general case)**.** *Assume that for some $\varepsilon > 0$, $(B^* M_B (B^*)^{-1}))^s - (1+\varepsilon) N_B$ is bounded from below and that $\tilde{\rho}_B$ is positive. Then for all $f \in \mathcal{C}_c^\infty(M)$, we have :*

$$\operatorname{Var}_\mu(f) \leq \frac{1}{\tilde{\rho}_B} \int_M |df|^2 \, d\mu.$$

Proof. Let $f \in \mathcal{C}_c^\infty(M)$ and $F_t = \mathbf{Q}_t^B \left((B^*)^{-1} df\right)$. As in (4.5.8), we set

$$\phi(t) = \int_M |F_t|_B^2 \, d\mu \tag{4.6.6}$$

and we have the following representation of the variance

$$\operatorname{Var}_\mu(f) = \int_0^{+\infty} \phi(t) \, dt. \tag{4.6.7}$$

We have:

$$\begin{aligned}
\phi'(t) &= 2 \int_M \left\langle F_t, L^{W,B} F_t \right\rangle_B dt \\
&\leq -2 \int_M \left\langle B^* F_t, \left[\left(B^* M_B (B^*)^{-1}) \right)^s - N_B \right] B^* F_t \right\rangle d\mu \\
&\leq -2 \tilde{\rho}_B \phi(t)
\end{aligned}$$

So we have

$$\phi(t) \leq e^{-2\tilde{\rho}_B t} \int_M |df|^2 \, d\mu. \tag{4.6.8}$$

Integrating on $\mathbb{R}_+$ gives the results. □

As for Theorem 4.5.3, the result still make sense when $\tilde{\rho}_B = 0$. With the same kind of hypothesis ,we can also prove a generalized Brascamp-Lieb inequality.

Theorem 4.6.3 (Generalized Brascamp-Lieb inequality - General case)**.** *Assume that $(B^* M_B (B^*)^{-1}))^s - (1+\varepsilon) N_B$ is bounded from below for some $\varepsilon > 0$ and that $(B^* M_B (B^*)^{-1}))^s - N_B$ is positive definite, then for every $f \in \mathcal{C}_c^\infty(M)$ we have :*

$$\operatorname{Var}_\mu(f) \leq \int_M \left\langle df, \left[\left(B^* M_B (B^*)^{-1}) \right)^s - N_B \right]^{-1} df \right\rangle d\mu. \tag{4.6.9}$$

Proof. First, let assume that $\tilde{\rho_B}$ is positive. Equation (4.6.2) implies that for all 1-form α we have:

$$\int_M \langle (-L^{W,B})\alpha, \alpha\rangle_B \, d\mu \geq \tilde{\rho_B} \int_M |\alpha|_B^2 \, d\mu. \tag{4.6.10}$$

As in the proof of Theorem 4.5.3, $-L^{W,B}$ is invertible with the same integral representation. So

$$\mathrm{Var}_\mu(f) = \int_M \left\langle (B^*)^{-1} df, (-L^{W,B})^{-1} \left((B^*)^{-1} df \right) \right\rangle_B \, d\mu. \tag{4.6.11}$$

Furthermore, we have:

$$\begin{aligned}\int_M \langle \alpha, (-L^{W,B})\alpha\rangle_B \, d\mu &\geq \int_M \left\langle B^*\alpha, \left[\left(B^* M_B (B^*)^{-1} \right)^s - N_B \right] B^*\alpha \right\rangle \, d\mu \\ &\geq \int_M \left\langle \alpha, (B^*)^{-1} \left[\left(B^* M_B (B^*)^{-1} \right)^s - N_B \right] B^*\alpha \right\rangle_B \, d\mu.\end{aligned}$$

As $(B^*)^{-1} \left[(B^* M_B (B^*)^{-1})^s - N_B \right] B^*$ is symmetric with respect to $\langle \cdot, \cdot \rangle_B$ and positive by assumption, we can use Lemma 4.5.1 to obtain

$$\mathrm{Var}_\mu(f) \leq \int_M \left\langle (B^*)^{-1} df, (B^*)^{-1} \left[\left(B^* M_B (B^*)^{-1} \right)^s - N_B \right]^{-1} B^* \left((B^*)^{-1} df \right) \right\rangle_B \, d\mu.$$

Now, if $\tilde{\rho}_B = 0$, we regularize as in the proof of Theorem 4.5.3. It ends the proof. □

We also obtain a bound for the spectral gap.

Proposition 4.6.4. *Assume that* $(B^* M_B (B^*)^{-1}))^s - (1+\varepsilon) N_B$ *is bounded from below for some* $\varepsilon > 0$ *and that* $\tilde{\rho}_B$ *is positive. Then the spectral gap* $\lambda_1(-L, \mu)$ *satisfies:*

$$\lambda_1(-L, \mu) \geq \tilde{\rho}_B. \tag{4.6.12}$$

Remark that if the hypothesis of Proposition 4.4.1 are satisfied, then $N_B = 0$ and $\rho_B = \tilde{\rho}_B$. In particular, Theorem 4.6.2 (respectively 4.6.3 and 4.6.4) can be interpreted as stability to small perturbations of the condition $\mathcal{B} = 0$. The bounds obtained are stable with respect to perturbations.

We finish with a toy example which illustrate the use of N_B when $\mathcal{B}$ does not vanish. Let $\theta : \mathbb{R}^2 \to \mathbb{R}$ a smooth function and let us define the twist

$$B^* = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

The associated tensor $\mathcal{B}$ satisfies :

$$\mathcal{B} = -2\nabla B \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

As it does not vanish, we must compute the correction term :

$$N_B = |\nabla\theta|^2 \,\mathrm{id}\,. \tag{4.6.13}$$

The twisting term is :

$$(B^*)^{-1} L^{/\!/}(B^*) = L(\theta) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - |\nabla\theta|^2 \,\mathrm{id}\,.$$

Its symmetric part is exactly compensated by N_B. Thus, for rotation, $\tilde{\rho}_B = \rho$. This phenomenon can be interpreted as the following : using only rotations, we do not change the metric. So it is impossible to improve the contraction properties of W^B.

4.7 Examples in radially symmetric surfaces

In this section, we illustrate our results with three examples. Each one corresponds to a different case where Bakry-Émery criterion is not satisfied : strictly convex in each point but $\rho = 0$, strictly concave in a compact region and a not upper bounded ρ, and strictly concave in a compact region and ρ upper bounded. We also give heuristic ideas to find adequate twists. We even improve a lower bound in a classical example. The difficulty is to find a concession between interesting examples (manifold and measure) and calculability. In our examples, the measure μ will be classical but the manifold will be from casual (as hyperbolic plan) to quite exotic. Our manifold M will be a two dimensional radially symmetric space with global polar chart $(r, \theta) \in \mathbb{R}_+ \times \mathbb{S}^1$, such that, in this coordinates, the metric is given by :

$$ds^2 = dr^2 + f(r)^2 d\theta^2 \tag{4.7.1}$$

where $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a smooth function satisfying $f(r) = 0$ if and only if $r = 0$ and $f'(0) = 1$. This model includes the surfaces of constant curvature : hyperbolic plan $\mathbb{H}^2$ with $f(r) = \sinh(r)$ or Euclidean plan $\mathbb{R}^2$ with $f(r) = r$. It also includes surfaces of revolution. The Riemanian volume measure of such a manifold is : $\mathrm{vol}(drd\theta) = f(r)drd\theta$. For every smooth function ϕ, in the orthonormal basis $(\partial_r, \frac{1}{f(r)}\partial_\theta)$, we have :

$$\nabla\phi = \begin{pmatrix} \partial_r \phi \\ \dfrac{1}{f}\partial_\theta \phi \end{pmatrix}, \tag{4.7.2}$$

$$\nabla^2\phi = \begin{pmatrix} \partial_r^2\phi & \frac{1}{f}\partial^2_{r,\theta}\phi - \frac{f'}{f^2}\partial_\theta\phi \\ \frac{1}{f}\partial^2_{r,\theta}\phi - \frac{f'}{f^2}\partial_\theta\phi & \frac{1}{f^2}\partial_\theta^2\phi + \frac{f'}{f}\partial_r\phi \end{pmatrix}, \tag{4.7.3}$$

$$\mathrm{Ric} = -\frac{f''}{f}\,\mathrm{id}\,. \tag{4.7.4}$$

For more details on radially symmetric manifold see [46], [58] or [72]. In all our examples, twists have the form of Proposition 4.4.2 : $B(x) = b(x)\,\mathrm{id}_{T_xM}$ with $b = \exp(U)$ a radial positive function. With this special form of twist, we have :

$$(B^*)^{-1}L^{/\!/}(B^*) = b^{-1}L(b)\,\mathrm{id}_{T_xM}, \tag{4.7.5}$$

and

$$b^{-1}L(b) = \Delta U - \langle \nabla V, \nabla U\rangle + |\nabla U|^2. \tag{4.7.6}$$

As in the Euclidean case, a usual choice of twist is $U = \varepsilon V$ but we will also see some cases where it is not enough to obtain a positive ρ_B. As metrics, measures and twists are radial, we will only use the variable r with a slight abuse of notation.

Our first example is the case of generalized Cauchy measures on $\mathbb{R}^2$. It have been studied in [68] for weighted Poincaré inequalities and in [19] for bounds on the spectral gap, both in any dimension $n \geq 2$. We show that our method can improve the previous lower bounds for $n = 2$. This example also illustrate how using Riemannian geometry can help for measures in an Euclidean space. On $\mathbb{R}^2$, we define the function $\sigma^2(x) = 1 + |x|^2$. For $\beta > 1$, we define the differential operator L_β by :

$$L_\beta f(x) = \sigma^2(x)\Delta_E f(x) - 2(\beta-1)x.\nabla_E f(x),\ \forall f \in \mathcal{C}^\infty(\mathbb{R}^2),\ \forall x \in \mathbb{R}^2, \tag{4.7.7}$$

where Δ_E and ∇_E stand for the Euclidean Laplacian and gradient. This operator admits a unique invariant probability μ_β whose density with respect to the Lebesgue measure is proportional to $(\sigma^2)^{-\beta}$. Remark that for $\beta \leq 1$, it does not define a finite measure any more. The form of the generator L_β suggests that the Euclidean geometry is not adapted to the problem. The relevant space is the manifold M which have a global Cartesian chart $\mathbb{R}^2$ and whose metric is given by

$$ds^2 = \frac{dx_1^2 + dx_2^2}{\sigma^2}.$$

In order to obtain an expression of the metric as in (4.7.1), we use the appropriate generalized polar coordinates :

$$(x_1, x_2) = (\sinh(r)\cos(\theta), \sinh(r)\sin(\theta))\,,\ (r,\theta) \in \mathbb{R}_+^* \times \mathbb{S}^1. \tag{4.7.8}$$

In the chart (r,θ), the metric has the desired form, with $f = \tanh$. Geometrically, we work on the revolution surface

$$\mathcal{S} = \left\{(\tanh(r)cos(\theta), \tanh(r)sin(\theta), g(r)); (r,\theta) \in \mathbb{R}_+ \times \mathbb{S}^1\right\}, \tag{4.7.9}$$

with $g(r) = \operatorname{argsinh}(\cosh(r)) - \cosh^{-2}(r)\sqrt{\cosh^2(r) + 1}$. It is a regularisation of a cylinder with a closed end (see Figure 4.1).

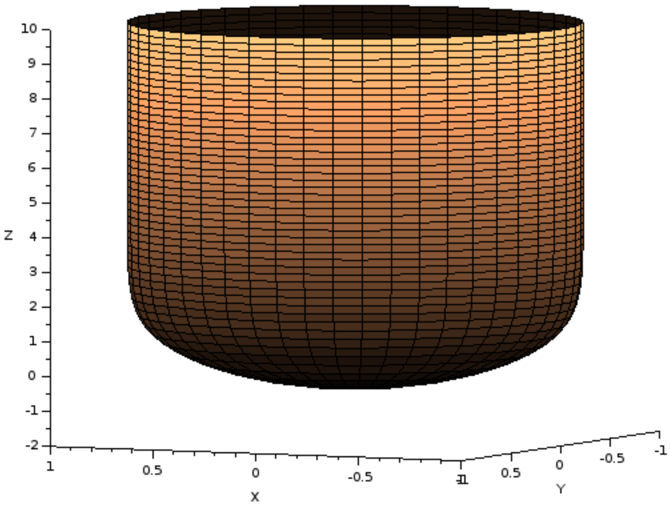


Figure 4.1: Riemannian surface associated to $\sigma^2\Delta_E$.

Then, we need to find the density of μ_β with respect to the Riemannian volume. We have :

$$\begin{aligned}
d\mu_\beta &= Z(\sigma^2)^{-\beta} dxdy \\
&= Z \cosh^{-2\beta} \sinh \cosh drd\theta \\
&= Z \cosh^{-2(\beta-1)} \tanh drd\theta \\
&= Z \cosh^{-2(\beta-1)} \operatorname{vol}(drd\theta),
\end{aligned}$$

with Z the normalization constant. Then, if we set $V(r) = 2(\beta - 1)\ln(\cosh(r))$ for $(r, \theta) \in \mathbb{R}_+$, the generator L_β has the expression (3.2.1) and we can apply our method. Firstly, the operator $\mathcal{M}$ is an homothetic transformation :

$$\mathcal{M} = \frac{2\beta}{\sigma^2} \operatorname{id}. \tag{4.7.10}$$

For each $x \in M$, $\mathcal{M}$ is strictly convex but globally, $\mathcal{M}$ is only convex : $\rho = 0$. We try a twist of the shape $\exp(\varepsilon V)$. Using the formula (4.7.6), for all $r \geq 0$, we have :

$$\rho_B(r) = 2\beta - 4\varepsilon(\beta-1) + [4\varepsilon(1-\varepsilon)(\beta-1)^2 - (2\beta - 4\varepsilon(\beta-1))]\tanh^2(r). \quad (4.7.11)$$

The function ρ_B is monotonous and can be bounded from below by the minimum between its value in $r = 0$ and its limit as $r \to +\infty$:

$$\rho_B \geq \min\{2\beta - 4\varepsilon(\beta-1), 4\varepsilon(1-\varepsilon)\}. \quad (4.7.12)$$

The optimal parameter is :

$$\varepsilon_0 = \begin{cases} \frac{1}{2} & \text{if} \quad 1 < \beta \leq 1+\sqrt{2} \\ \frac{\beta+\sqrt{(\beta-1)^2-1}}{2(\beta-1)} & \text{if} \quad 1+\sqrt{2} \leq \beta \end{cases} \quad (4.7.13)$$

.

Corollary 4.7.1. *The spectral gap of the operator L_β is bounded from below by :*

$$\lambda_1(L_\beta) \geq \begin{cases} (\beta-1)^2 & \textit{if} \quad 1 < \beta \leq 1+\sqrt{2} \\ 2\sqrt{(\beta-1)^2-1} & \textit{if} \quad 1+\sqrt{2} \leq \beta \end{cases}. \quad (4.7.14)$$

Back to $\mathbb{R}^n$, this spectral gap is interpreted as weighted Poincaré inequality :

$$\mathrm{Var}_\mu(f) \leq \frac{1}{\rho_B}\int_M |\nabla_E f|^2 \sigma^2 d\mu_\beta, \ \forall f \in \mathcal{C}_0^\infty(M).$$

Remark that for $\beta \geq 1+\sqrt{2}$, the optimal ε_0 corresponds to the case where ρ_B is a constant function. The best lower bound known for this spectral gap, in [19], are :

$$\lambda_1(L_\beta) \geq \begin{cases} (\beta-1)^2 & \text{if} \quad 1 < \beta \leq \frac{3+\sqrt{5}}{2} \\ \beta & \text{if} \quad \frac{3+\sqrt{5}}{2} \leq \beta \end{cases}.$$

So our method improves the result for $\beta \geq \frac{3+\sqrt{5}}{2}$. Actually, [19] also gives upper bounds :

$$\lambda_1(L_\beta) \leq \begin{cases} (\beta-1)^2 & \text{if} \quad 1 < \beta \leq \frac{3+\sqrt{5}}{2} \\ 2(\beta-1) & \text{if} \quad \frac{3+\sqrt{5}}{2} \leq \beta \end{cases},$$

and for $\beta \geq 3$, it is proved in [68] that $\lambda_1(L_\beta) = 2(\beta-1)$. This shows that our lower bound is optimal for $\beta \leq 1+\sqrt{2}$ and has the good asymptotic for $\beta \to +\infty$, even if our choice of twist, a priori, cannot pretend to be optimal.

Our second example is the case of exponential power measures on the hyperbolic plan. We set $M = \mathbb{H}^2$, $f = \sinh$ and for $\alpha > 2$,

$$V(r) = \frac{r^\alpha}{\alpha}, \ \forall r \in \mathbb{R}_+.$$

Remark that the measure associated to V is finite for $\alpha > 1$. The generator associated to this measure is

$$L_\alpha = \partial_r^2 + \frac{1}{\tanh(r)}\partial_r + \frac{1}{\sinh^2(r)}\partial_\theta^2 - r^{\alpha-1}\partial_r. \tag{4.7.15}$$

Using a result from [76], these measures satisfy a Log-Sobolev inequality for $\alpha \geq 2$. The limit case $\alpha = 2$, corresponds to the radial hyperbolic Ornstein-Uhlenbeck process. We will discuss at the end why it must be excluded by our method. The smallest eigenvalue of the potential $\mathcal{M}$ is :

$$\rho_\alpha(r) = \min\left\{(\alpha-1)r^{\alpha-2}, \frac{r^{\alpha-1}}{\tanh(r)}\right\} - 1, \, \forall r \in \mathbb{R}_+, \tag{4.7.16}$$

so its infimum is $\rho = -1$. Then, we know that the semi-groups $\mathbf{P}$ and $\mathbf{Q}$ are intertwined but as ρ is not positive, we cannot directly use it in terms of functional inequalities. It is a case where twisting is needed. In these cases, the operator $\mathcal{M}$ is concave in a neighbourhood of the origin and strictly convex outside . We need a choice of b which counter the negativity of Ric around $r = 0$. A direct calculation show that $U = \varepsilon V$ cannot achieve this goal. We propose the following function :

$$U_{\varepsilon,\eta}(r) = \frac{1}{2}V(r) - \varepsilon\frac{r^2}{2} + \eta\ln(\cosh(r)), \, \forall r \in \mathbb{R}_+,$$

with ε and η parameters which should be fitted. Let us explain this choice. In the expression (4.7.6) there is the beginning of the square $|\nabla U - \nabla V/2|^2$ which explains the term $V/2$. The second term is inspired by the one-dimensional case in [18]. Its Laplacian should help in $r = 0$ because it will not vanish there. The third term is directly linked to the metric : it is a primitive of f/f'. This makes appear the Ricci curvature in the expression of $b^{-1}L(b)$. With this choice of U, for all $r \in \mathbb{R}_+$, we have :

$$\begin{aligned} b^{-1}L_\alpha(b)(r) = 2\eta - \varepsilon + \frac{\alpha-1}{2}r^{\alpha-2} + \varepsilon^2 r^2 - \frac{r^{2\alpha-2}}{4} + \frac{r^{\alpha-1} - 2\varepsilon r}{2\tanh(r)} \\ - 2\varepsilon\eta r\tanh(r) - \eta(1-\eta)\tanh^2(r). \end{aligned}$$

Now, we need to find whether there exists coefficient ε and η such that ρ_B is positive and for which coefficient it is optimal. Unfortunately, it seems difficult to give explicit bounds in all generality. Nevertheless, numerically, we find the following bounds :

α	2.01	3	3.5	4	4.5
(ε, η)	$(0.5, -0.007)$	$(1.59, 0.8)$	$(1.83, 1.15)$	$(2.45, 1.84)$	$(2.78, 2.279)$
$\lambda_1(L_\alpha) \geq \cdot$	6.10^{-4}	0.47	0.34	0.21	2.10^{-3}

Remark that the method developed in [19] needs $\alpha > 3$ to obtain results. These bounds do not pretend to be optimal. Perhaps another kind of b could have brought better results, especially for $\alpha = 4.5$, or $\alpha = 5$ for which we did not find good parameters. It could be interesting to bring some upper-bound on the spectral gap, by a testing on examples or by other means, to discuss the relevancy of our lower-bound. Concerning the decay of our bounds for α near 2, it was expected. Indeed, for $\alpha = 2$, ρ_2 is a constant function, equal to zero. But as explained in [2], an integration by parts shows that

$$\int_M -b^{-1} L(b)\, d\mu = -\int_M |db|^2\, d\mu. \tag{4.7.17}$$

If b is not constant, it will be negative in a region and so ρ_B. Twisting with a function absolutely needs a region where $\mathcal{M}$ is strictly convex. It is not the case of the hyperbolic Ornstein-Uhlenbeck generator. It could be interesting to look at more complex B, in the way of Section 4.6.

In our last, example, we have a similar situation : bounded region of strict concavity and strict convexity elsewhere but with another constraint : ρ is upper bounded. Let M be the revolution surface of Figure 4.2 generated by the rotation around the ordinates axis of the curve

$$y = \frac{1}{\sqrt{1+x^2}},\ x \in \mathbb{R}_+. \tag{4.7.18}$$

It is a regularized version of the surface generated by $y = 1/x$. In an adapted polar chart, its metric has the form (4.7.1) but we don't have any explicit formula for f. However, we can find sufficiently sharp estimates of it. For a surface in $\mathbb{R}^3$ parametrized as

$$S = \left\{ (f(r)\cos(\theta), f(r)\sin(\theta), g(r)) : (r,\theta) \in \mathbb{R}_+ \times \mathbb{S}^1 \right\}, \tag{4.7.19}$$

the metric has the form : $ds^2 = (f'^2 + g'^2)dr^2 + f^2 d\theta^2$. Using the relation between f and g given by the generating curve, we can prove that f satisfies the equation :

$$f' = \frac{1}{\sqrt{1 + \frac{f^2}{(1+f^2)^3}}}.$$

We obtain the following bounds : for all $r \in \mathbb{R}_+$,

$$\alpha r \leq f(r) \leq r$$
$$\frac{1}{\sqrt{1 + \frac{r^2}{(1+\alpha^2 r^2)^3}}} \leq f'(r) \leq \frac{1}{\sqrt{1 + \frac{\alpha^2 r^2}{(1+r^2)^3}}}$$

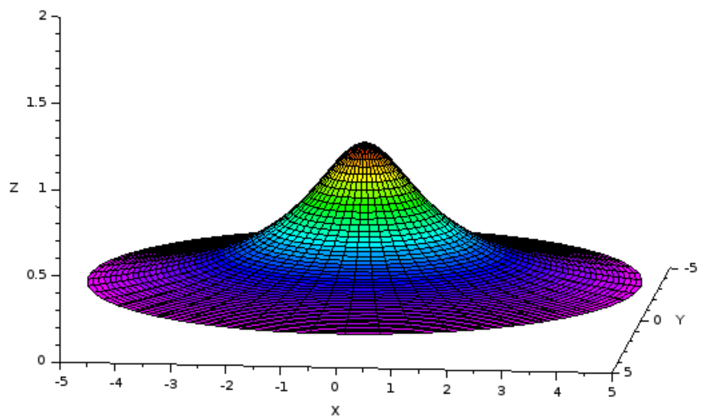


Figure 4.2: Revolution surface associated to $y = \frac{1}{\sqrt{x^2+1}}$.

,

with $\alpha = \sqrt{\frac{27}{31}}$. The Ricci curvature has an explicit formula in function of f :

$$\mathrm{Ric} = \frac{(1-2f^2)(1+f^2)^2}{(f^2+(1+f^2)^3)^2}. \tag{4.7.20}$$

This formula gives us lower and upper bounds on Ric. In particular, we know that $\mathrm{Ric}(0) = 1$, then it decreases to a negative minimum (which is numerically in the range $-0.050 < \min \mathrm{Ric} < -0.049$) and then it increases and vanishes (see Figure 4.3).

We are interested in radial Gaussian measures on this manifold : for $\gamma > 0$,

$$V_\gamma(r) = \frac{\gamma r^2}{2}, \ \forall r \in \mathbb{R}_+.$$

The smallest eigenvalue of $\mathcal{M}$ is :

$$\rho(r) = \gamma \min\left\{1, r\frac{f'(r)}{f(r)}\right\} + \mathrm{Ric}(r), \ \forall r \geq 0.$$

If $\alpha\gamma$ is bigger to $-\min \mathrm{Ric}$, ρ will be positive. So we are mainly interested in the case of small γ such that twistings are unavoidable, but also in cases of "big" γ

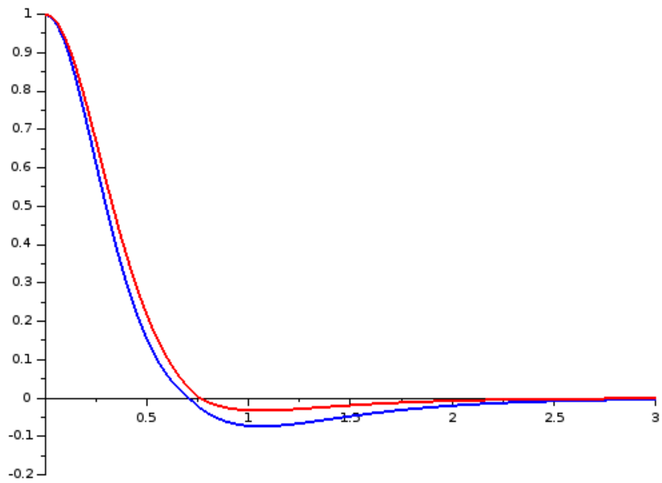


Figure 4.3: Upper-bound (in red) and lower bound (in blue) of Ricci curvature.

as we shall see. Thanks to the Ricci curvature, we know that $\rho(r)$ is positive for small r and tends to $\gamma > 0$ as r tends to $+\infty$. We need to compensate a compact region of negativity. Here, we use the radial function $U_{\varepsilon,\omega,k}$ defined by

$$U_{\varepsilon,\omega,k}(r) = \int_0^r \varepsilon \sin(\omega t) e^{-kt}\, dt,\ \forall r \in \mathbb{R}_+, \tag{4.7.21}$$

where ε, ω and k are parameters. This goal of this quite unusual function is to give to $b^{-1}L(b)$ the shape of Ric. The decreasing exponential term is explained by the vanishing of Ric, it is linked to the boundedness of ρ. The goal of the sinusoidal term is to create a peak, compensating Ricci's minimum. The exponential has to damp the following peaks. According to equation (4.7.6), for all $0 \leq r$, we have :

$$b^{-1}L(b)(r) = \varepsilon \left[\omega \cos(\omega r) + \sin(\omega r) \left(-k + \frac{f'(r)}{f(r)} - \gamma r + \varepsilon \sin(\omega r) e^{-kr}\right)\right] e^{-kr}. \tag{4.7.22}$$

Again, we can obtain numerical lower bound of the spectral gap. With the parameters $\varepsilon = 0.217$, $\omega = 2.022$, $k = 1.7$, we have :

γ	0.01	0.02	0.03	0.04	0.06
$\lambda_1(L_\gamma) \geq \cdot$	7.10^{-3}	0.01	0.02	0.03	0.05

As expected, this bound are smaller than γ, the Euclidean bound, although they do not seem very sharp, in particular, for very small γ. It could be linked to the choice of twist b. When γ become smaller, the region where $\mathcal{M}$ is strictly concavity become larger. This explains why our choice of twist is less adapted for small γ. To finish, this example shows that twisting is not the last resort method of spectral gap research and can also be interesting even if the Backy-Émery criterion is satisfied. Looking at $\alpha\gamma$ slightly bigger than $-\min \mathrm{Ric}$, the upper bound $\rho \leq \gamma + \min \mathrm{Ric}$ of Bakry-Émery is very small but as shown in the array above, we still obtain reasonable bounds ($\gamma = 0.06$). For $\gamma = 1$, where we have more room for our parameters, we can obtain a lower bound $\lambda_1(L) \geq 0.98$ (here, we use $k = 2.5$) while ρ is in the range $0.95 < \rho < 0.951$.

Chapter 5

Intertwining and preservation of positivity

In this chapter we present some work around FKG inequality. The main result is a criterion of positivity preservation by some deformed parallel translation on Lie groups. This work does not have lead to a publication yet.

Contents

5.1 Introduction

This chapter is motivated by the article [13] about the Fortuin-Kastelyn-Ginibre inequality, or FKG inequality. Unlike the inequalities seen in Chapter 3 and Chapter 4, which give upper bound of the covariance, the FKG inequality states the non-negativity of it. This inequality comes from the study of Gibbs measures on spin system. It was introduced in [39] for partially ordered lattices. In [13], the authors introduce partial differentials on lattices. Their proof consists on studying the action of these differentials on a symmetric semi-group. Without its vocabulary nor its formalism, the core of their proof is an intertwining. Besides, the introduction of differentials transpose the notion of increasing functions, to the notion of positive vectors more extendible to manifolds. Semi-groups preserving increasing functions are intertwined to semi-groups preserving positive vectors. It allows to extend the FKG inequality to different kind of spaces. In [13], this is illustrated in the Euclidean space $\mathbb{R}^n$. In this work, we tried to extend it to Riemannian manifolds. The notion of partial derivative can be adapted if we have globally defined frames of the tangent space. This naturally leads to the study of Lie groups.

In the fist section, we present the FKG inequality in $\mathbb{R}^n$ and we bring a new proof of the Euclidean result. In Section 5.3, we prove that the Brownian semi-group in a Lie group preserves the monotonicity of functions. It allows to understand which notion of increasing functions we should chose. This is illustrated with the Poincaré half-plan. In Section 5.4, we present a remarkable diffusion on the Heisenberg group. The proofs are not very adaptable but enlighten some expected properties of the result for general diffusions and make a link with the Euclidean case. In Section 5.5, we introduced a parallel translation above diffusion, with generator $\frac{1}{2}\Delta + U$, on Lie group, with U a vector field. We investigate its properties and similarities to the classical deformed parallel translation. We find a criterion on the potential U to have the preservation of positive vector field and we give some examples. This work have not been entirely investigated yet. We present in Section 5.6 some points which seems interesting to us.

5.2 FKG inequality on $\mathbb{R}^n$

On the euclidean space $\mathbb{R}^n$, let μ be a probability measure defined by

$$\mu(dx) = e^{-V(x)}dx,$$

where V is a smooth potential with $\nabla^2 V$ bounded from below. To the measure μ, is associated the semi-groups $\mathbf{P}$, with generator $L = \Delta - \nabla V.\nabla$ and $\mathbf{Q}$ the semi-group on 1-forms, of the deformed parallel translation, with generator L^W. According to the Bakry-Émery criterion, we have the stochastic representation of $\mathbf{Q}$ on bounded 1-forms and these semi-groups are intertwined. We denote by $\mathcal{C}_b^1(\mathbb{R}^n)$ the space of $\mathcal{C}^1$ functions, bounded with bounded derivatives. It is a good space to deal with the intertwining : for $f \in \mathcal{C}_b^1(\mathbb{R}^n)$, $\mathbf{P}f$ and $\mathbf{Q}(\nabla f)$ are well-defined and we have :

$$d\mathbf{P}_t f = \mathbf{Q}_t df. \tag{5.2.1}$$

Before presenting the FKG inequality, we need some vocabulary. A vector is said non-negative (respectively, positive) if all its coordinates, in the canonical basis, are non-negative (respectively, positive). For $x, y \in \mathbb{R}^n$, we denote $x \leq y$ (respectively $x < y$) if $y - x$ is non-negative (respectively, positive). A real-valued function f is said increasing if it verifies :

$$x < y \Rightarrow f(x) < f(y). \tag{5.2.2}$$

The definition of non-decreasing, decreasing or monotonous are understood the same way. For smooth functions, these properties can be characterised by their gradient : a function f is increasing if and only if ∇f is positive.

Definition 5.2.1. *A measure ν on $\mathbb{R}^n$ satisfies the FKG property if for all non-decreasing $f, g \in L^2(\nu)$, we have :* $\mathrm{Cov}_\nu(f,g) \geq 0$.

The previous correlation inequality is, strictly speaking, the FKG inequality. We want to find a criterion on the potential to satisfy FKG. Such a condition have been proved in [13]. We propose here a new proof, based on intertwining and deformed parallel translation.

Theorem 5.2.2. *Assume that V satisfies $\nabla^2 V \geq \rho > -\infty$ and $\partial^2_{ij} V \leq 0$ for $i \neq j$. Let f in $\mathcal{C}^1_b(M)$ be increasing, then for all $t \geq 0$, $\mathbf{P}_t f$ is increasing.*

Proof. According to the intertwining relation

$$\nabla \mathbf{P}_t f = \mathbf{Q}_t(\nabla f), \, \forall t \geq 0, \tag{5.2.3}$$

it is enough to show that $\mathbf{Q}$ preserve the non-negativity of ∇f. For all $t \geq 0$:

$$\mathbf{Q}_t(\nabla f) = \mathbb{E}\left[\langle \nabla f, W_t \cdot \rangle\right]. \tag{5.2.4}$$

Then, it is enough to prove that for all $v \geq 0$, we have $W_t v \geq 0$. We denote $(e_k)_k$ the canonical basis of $\mathbb{R}^n$ and we fix v a positive vector. By definition of the deformed parallel translation, for all $1 \leq i \leq n$, we have :

$$\begin{aligned} d\langle W_t v, e_i \rangle &= -\nabla^2 V(W_t v, e_i) dt \\ &= -\partial^2_i V(X_s) \left\langle W^X_t v, e_i \right\rangle dt - \sum_{j \neq i} \partial^2_{ij} V(X_s) \left\langle W^X_t v, e_j \right\rangle dt \end{aligned}$$

The initial condition is : $W_0 v = v > 0$. Let τ be the first time when a coordinate of $W_t(v)$ become negative :

$$\tau = \inf\left\{t \geq 0 / \exists k, \langle W_t(v), e_k \rangle < 0\right\}. \tag{5.2.5}$$

If τ is finite, then there is an index i_0 so that : $\langle W_\tau(v), e_{i_0} \rangle = 0$. Yet, we have,

$$\langle W_\tau v, e_{i_0} \rangle = e^{A(\tau)} v_{i_0} - e^{A(\tau)} \int_0^\tau e^{-A(s)} \sum_{j \neq i_0} \underbrace{\partial^2_{i_0 j} V(X_s)}_{\leq 0} \underbrace{\langle W^x_s v, e_j \rangle}_{\geq 0} ds > 0, \tag{5.2.6}$$

with

$$A(t) = \int_0^t -\partial_{i_0}^2 V(X_s) ds. \tag{5.2.7}$$

This is a contradiction. So, for all $t \geq 0$, $W_t v$ is positive. By continuity, $W_t v$ is non-negative for all $v \geq 0$ and so, for all $t \geq 0$, $\mathbf{Q}(\nabla f) \geq 0$ and $\mathbf{P}_t f$ is non-decreasing. □

Corollary 5.2.3. *Assume that V satisfies $\nabla^2 V \geq \rho > -\infty$ and $\partial_{ij}^2 V \leq 0$ for $i \neq j$, then μ satisfies FKG.*

Proof. First, lets assume that f and g are in $\mathcal{C}_b^1(\mathbb{R}^n)$. According to the covariance representation 3.3.1, we have :

$$\mathrm{Cov}_\mu(f, g) = \int_0^\infty \left(\int_{\mathbb{R}^n} \langle \nabla f, \mathbf{Q}_t(\nabla g) \rangle \, d\mu \right) dt. \tag{5.2.8}$$

As f is increasing, ∇f is positive. According to the previous theorem, $\mathbf{Q}_t(\nabla g)$ is also positive for all $t \geq 0$. Then f and g satisfy FKG. We conclude by an argument of density in $L^2(\mu)$. □

The core of this proof is the preservation of positivity by the deformed parallel translation. This property can be studied as long as the translation is defined, even if intertwining is not satisfied nor $\mathcal{C}^0$ semi-group defined. The intertwining relation allows to go from positivity preservation from monotonicity preservation and FKG inequality. These results are subordinated to a notion of positive vectors. In $\mathbb{R}^n$ this notion is canonical. In a manifold, this notion can fail to exist globally. It can be defined only in parallelizable manifolds. This is the reason why we will try to extend the results, in the following sections, to Lie groups.

5.3 Brownian motion on Lie groups.

In this section, we defined a notion of increasing functions on a Lie group and we prove a result of preservation of monotonicity by the Brownian semi-group. We give the example in the hyperbolic plan $\mathbb{H}^2$.

Let $(G, \langle \cdot, \cdot \rangle_g)$ be a Lie group, equipped with a left-invariant metric. For all $g \in G$, we denote by L_g and R_g the left-multiplication and right-multiplication by g. Left-multiplications are isometric transforms of G. If the metric is bi-invariant, R_g are also isometric but the existence of such a metric is a very strong assumption on the group (see [63]). Lie groups are parallelizable manifold : there exists globally defined vector fields, even if there is not any global chart. For example, let $(H_i)_i$ be an orthonormal basis of $\mathfrak{G}$. It is associated to a family of right-invariant vector fields $(H_i^R)_i$ defined by : $H_i^R(g) = dR_g.H_i$. This provide a moving frame on G and a notion of increasing functions.

Definition 5.3.1. *A function $f \in \mathcal{C}^1(M)$ is increasing along $X \in \Gamma(TG)$ if for all $g \in G$, $\langle df, X(g)\rangle > 0$. A function is increasing along a frame $(X_i)_i$ if it is increasing along each vector field X_i.*

Similarly, we can define the notion of non-decreasing, decreasing or monotonic functions.

In $\mathbb{R}^n$, a Brownian flow can be construct with only one Brownian motion starting from 0. It is possible to do so in a Lie group. Let $(B^i)_i$ be n independent real Brownian motions. The process $B_t = \sum H_i B_t^i$ is a Brownian motion in the Lie algebra $\mathfrak{G}$, seen as a vectorial space. We define the process $(X_t)_{t\geq 0}$ by the Ito equation:

$$\begin{cases} d^\nabla X_t = dL_{X_t} dB_t \\ \quad X_0 = e \end{cases} \tag{5.3.1}$$

and a flow by left-multiplication : $X_t(g) = L_g.X_t$.

Lemma 5.3.2. *The process $X_t(g)$ is a Brownian motion starting from g.*

Proof. For all $f \in \mathcal{C}_c^\infty(G)$, we have :

$$\begin{aligned} f\left(X_t(g)\right) - f\left(X_0(g)\right) &= \int_0^t \langle df, H_i^L(X_s(g))\rangle dB_s^i \\ &\quad + \frac{1}{2}\int_0^t \nabla^2 f(H_i^L, H_j^L)(X_s(g))\langle dB^i, dB^j\rangle_s \\ &\overset{(m)}{=} \frac{1}{2}\int_0^t \nabla^2 f(H_i^L, H_j^L)(X_s(g))ds \\ &\overset{(m)}{=} \frac{1}{2}\int_0^t \Delta f(X_s(g))ds \end{aligned}$$

So $X(g)$ is a diffusion with generator $\frac{1}{2}\Delta$, starting from g. □

We denote by $\mathbf{P}$ the associated semi-group. The flow $\left(X(g)\right)_{g\in G}$ give a simple proof for the preservation of monotonicity. We denote by $\mathcal{C}_b^1(G)$ the space of $\Re$-valued $\mathcal{C}^1$ bounded functions, with bounded differential.

Proposition 5.3.3. *Let $\hat{H}$ be a right-invariant vector field and $f \in \mathcal{C}_b^1(M)$ be increasing along $\hat{H}$, then for all $t \geq 0$, $\mathbf{P}_t f$ is also increasing along $\hat{H}$.*

Unlike the general case of diffusion in $\mathbb{R}^n$, this result only needs a monotonicity in along one vector field. Of course, this implies the preservation of monotonicity along any right-invariant frame.

Proof. Let $\hat{H}$ be a right-invariant vector field with $\hat{H}(e) = H \in \mathfrak{G}$. We have :

$$\begin{aligned}
\left\langle d\mathbf{P}_t f, \hat{H}(g) \right\rangle &= \left.\frac{d}{da}\right|_{a=0} \mathbf{P}_t f(\exp(aH)g) \\
&= \left.\frac{d}{da}\right|_{a=0} \mathbb{E}\left[f(\exp(aH)gX_t)\right] \\
&= \mathbb{E}\left[\langle df, dR_{gX_t}H\rangle\right] \\
&= \mathbb{E}\left[\langle df, \hat{H}(gX_t)\rangle\right] \\
&\geq 0
\end{aligned}$$

□

The core of this proof is the left-invariance of the law. For the diffusion with generator $\frac{1}{2}\Delta + U$, with U a left invariant vector field, we can also produce a flow by left translation. So the associated semi-group will also preserve monotonicity along right-invariant vector fields. To make a link with the previous section, the deformed parallel translation above the Brownian motion is $W_t = dR_{X_t}$.

Let us apply this result to a very classical manifold : the hyperbolic plan $\mathbb{H}^2$ with its usual metric. Its Poincaré half-plan model global chart $(x, y) \in \mathbb{R} \times \mathbb{R}_+^*$ induce the moving frame (∂_x, ∂_y). The hyperbolic Laplacian is given by

$$\Delta = y^2(\partial_x^2 + \partial_y^2).$$

We have an explicit expression of the associated Brownian flow : if B and W are independent real Brownian motions and $(x_0, y_0) \in \mathbb{R} \times \mathbb{R}_+^*$, then the process defined for $t \geq 0$ by

$$\begin{cases} X_t = x_0 + \displaystyle\int_0^t Y_s dW_s \\ Y_t = y_0 e^{B_t - t/2}, \end{cases} \tag{5.3.2}$$

is an hyperbolic Brownian motion starting from (x_0, y_0). We are looking for vector fields along which the monotonicity is preserved by the Brownian semi-group

$$\mathbf{P}_t f(x, y) = \mathbb{E}\left[f\left(x + y\int_0^t e^{B_s - s/2} dW_s, y e^{B_t - t/2}\right)\right], \ \forall (x, y) \in \mathbb{R} \times \mathbb{R}_+^*, \ t \geq 0. \tag{5.3.3}$$

The frame (∂_x, ∂_y) was not defined for any geometric reason. and an easy calculation show that the monotonicity is not preserved along ∂_y. Yet, it is preserved along ∂_x. We want to understand why and to complete the frame of monotonicity preserved directions. This is where we use our result on Lie groups. We consider $G = \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$. By the Iwasawa decomposition, G is isomorphic to NA where

$$A = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, a \in \mathbb{R}^* \right\} \quad \text{and} \quad N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, x \in \mathbb{R} \right\}. \tag{5.3.4}$$

As G acts freely and transitively on the hyperbolic plane, it is isomorphic to it. So there is a global map of $\mathbb{H}^2$ given by:

$$\begin{array}{ccccc} \mathbb{R} \times \mathbb{R}_+^* & \longrightarrow & NA & \longrightarrow & \mathbb{H}^2 \\ (x,y) & \mapsto & M = \begin{pmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix} & \mapsto & M.i = x + iy \end{array}$$

In this map, the right-invariant vector fields of $\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$ are :

$$X^R(x,y) = \partial_x \quad \text{and} \quad Y^R(x,y) = x\partial_x + y\partial_y. \tag{5.3.5}$$

The flow of X^R is the family of lines parallel to the x-axis. They are horocycles. The flow of Y^R is a family of rays starting from $(0,0)$ (see Figure 5.1). Remark that the left-invariant vector-fields are

$$X^L(x,y) = y\partial_x \quad \text{and} \quad Y^L(x,y) = y\partial_y, \tag{5.3.6}$$

and from an orthonormal frame for the hyperbolic metric, as expected.

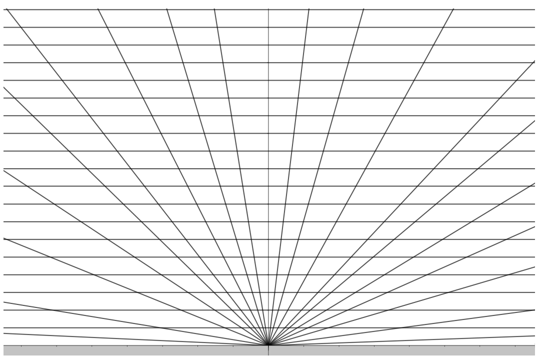
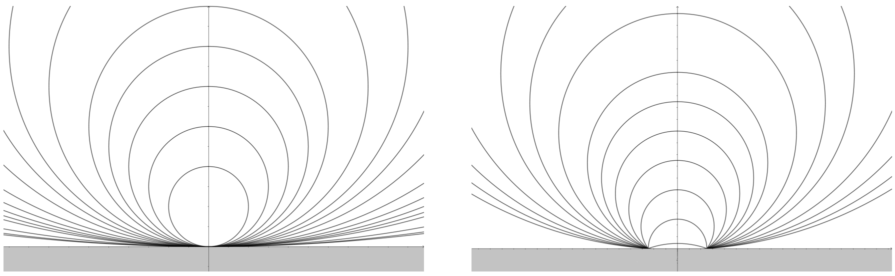


Figure 5.1: Grid on $\mathbb{H}^2$ by the vector fields $X^R = \partial_x$ and $Y^R = x\partial_x + y\partial_y$.

The frame (X^R, Y^R) is the good grid to study the monotonicity in $\mathbb{H}^2$. The functions $(x,y) \mapsto x/y$ and $(x,y) \mapsto y$ can be seen as elementary increasing functions. The semi-group of hyperbolic Brownian motion preserve the functions increasing along (X^R, Y^R). This could be verify easily, using the following stochastic representation of the semi-group (5.3.3). It explains why ∂_x is a vector field along which the monotonicity is preserved. Furthermore, acting with an homography does not change the Brownian law because they are isometric. So we can act on the frame (X^R, Y^R) and extend the class of functions whose monotonicity is preserved. The flow of X^R is transformed to a family of horocycles intersecting on the same point of the boundary : a family of lines parallel to the x-axis ore a family of circles tangent to it on a fixed point. The flow of Y is transformed in a family of ray starting from the same point on the x-axis or a family of circle crossing the x-axis in two fixed points (see figure 5.2).



(a) (b)

Figure 5.2: Transformation of the X^R flow (a) and Y^R flow (b) under isometry.

5.4 Interlude - On Heisenberg group.

In this section, we continue our study of monotonicity preserving semi-groups with the example of a particular diffusion on the Heisenberg group. As this diffusion does not admit any flow by left-multiplication, the proof of Proposition 5.3.3 is not adaptable. However, we manage to compute the action of the right-invariant vector field and obtain a result of monotonicity's preservation. We also derived a different kind of intertwining relation and we investigate its invariant law.

The Heisenberg group, $\mathcal{H}$, is the set $\mathbb{R}^3$ endowed with the following product $*$:

$$(x, y, z) * (\hat{x}, \hat{y}, \hat{z}) = \left(x + \hat{x}, y + \hat{y}, z + \hat{z} + \frac{1}{2}(x\hat{y} - y\hat{x})\right). \tag{5.4.1}$$

The first and second coordinates can be understood as distances and the third as an area. The term $\frac{1}{2}(x\hat{y} - y\hat{x})$ is interpreted as the algebraic area between the segment $[(0,0),(x+\hat{x}, y+\hat{y})]$ and the piecewise path of segments $[(0,0),(x,y)]$ and $[(x,y),(x+\hat{x}, y+\hat{y})]$ (see Figure 5.3). This group has a structure of Lie group with a global chart. An alternative definition of this group is the 3×3 upper-triangular matrices with the identity on the diagonal. For more details on the Heisenberg group, see [64] or [9].

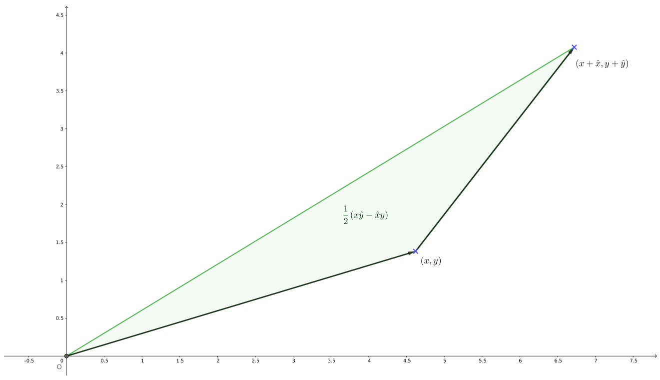


Figure 5.3: Illustration of Heisenberg group law

Its Lie algebra $\mathfrak{H}$ is spanned by ∂_x, ∂_y and ∂_z. We endow $\mathcal{H}$ with a left-invariant metric which makes it orthonormal. We consider the right-invariant and the left-invariant vector fields defined for all $g=(x,y,z)$, by :

$$\begin{cases} X^R(g) = \partial_x + \frac{1}{2} y \partial_z \\ Y^R(g) = \partial_y - \frac{1}{2} x \partial_z \\ Z^R(g) = \partial_z \end{cases}, \qquad \begin{cases} X^L(g) = \partial_x - \frac{1}{2} y \partial_z \\ Y^L(g) = \partial_y + \frac{1}{2} x \partial_z \\ Z^L(g) = \partial_z \end{cases} \tag{5.4.2}$$

According to Proposition 5.3.3, the monotonicity along X^R, Y^R or Z^R are preserved by the Brownian semi-group. We are interested in an other diffusion on $\mathcal{H}$, linked to its sub-Riemannian structure. Among the automorphisms of $\mathcal{H}$, the family of dilations δ_λ, for $\lambda \in \mathbb{R}$, is defined by :

$$\delta_\lambda(x,y,z) = (\lambda x, \lambda y, \lambda^2 z), \ \forall (x,y,z) \in \mathbb{R}^3. \tag{5.4.3}$$

We define the two following differential operators :

$$L = \frac{1}{2}\left(X^{R^2} + Y^{R^2}\right), \qquad D = \frac{1}{2}\left(x\partial_x + y\partial_y + 2z\partial_z\right). \tag{5.4.4}$$

The operator L is the sub-Riemannian Laplacian on $\mathcal{H}$ and D is the dilation operator : dilation automorphisms can be seen as the semi-group of D :

$$D_g = \frac{1}{2} \left.\frac{d}{dt}\right|_{t=1} \delta_{e^t}(g), \ \forall g \in \mathcal{H}. \tag{5.4.5}$$

Let B end W be two independent real Brownian motions. For $g \in \mathcal{H}$, we define the two following process:

$$\begin{cases} \circ dG_t(g) = X^R(G_t(g)) \circ dB_t + Y^R(G_t(g)) \circ dW_t \\ \quad G_0(g) = g \end{cases}, \tag{5.4.6}$$

$$\begin{cases} \circ d\hat{G}_t(g) = X^R(\hat{G}_t(g)) \circ dB_t + Y^R(\hat{G}_t(g)) \circ dW_t - \alpha D(\hat{G}_t(g))dt \\ \quad \hat{G}_0(g) = g \end{cases} \tag{5.4.7}$$

Proposition 5.4.1. *The process G is a flow of the sub-Riemannian Brownian motion and $\hat{G}$ is a flow of the diffusion with generator $L - \alpha D$.*

Remark that D is not the gradient of any function so $\hat{G}$ is not a reversible diffusion. We denote by $\mathbf{P}$ and $\hat{\mathbf{P}}$ the semi-groups associated to G and $\hat{G}$ respectively. The flow $G(g)$ is a left-multiplicative flow. This means that, as for Proposition 5.3.3, $\mathbf{P}$ preserve the monotonicity along X^R, Y^R or Z^R. In the case of $\hat{G}$, we cannot use this trick : the law of $\hat{G}(g)$ is different from the law of $L_g\hat{G}(e)$. Yet, it is possible work with an explicit expression of the flow.

Lemma 5.4.2. *For $g = (x_0, y_0, z_0) \in \mathcal{H}$ and $t \geq 0$, the coordinates $(\hat{x}_t, \hat{y}_t, \hat{z}_t)$ of $\hat{G}_t(g)$ are given by*

$$\begin{cases} \hat{x}_t = e^{-\frac{\alpha t}{2}} \left(x_0 + \int_0^t e^{\frac{\alpha s}{2}} dB_s \right) \\ \hat{y}_t = e^{-\frac{\alpha t}{2}} \left(y_0 + \int_0^t e^{\frac{\alpha s}{2}} dW_s \right) \\ \hat{z}_t = e^{-\alpha t} \left(z_0 + \int_0^t e^{\alpha s} dM_t^{g_0} \right) \end{cases}$$

where the semi-martingales M_t^g are defined by : $M_t^g = \frac{-\hat{y}_t}{2} dB_t + \frac{\hat{x}_t}{2} dW_t$.

Proof. Rewriting the equation (5.4.7) with the coordinates, we have :

$$\begin{cases} \circ d\hat{x}_t = \circ dB_t - \dfrac{\alpha \hat{x}_t}{2} dt \\ \circ d\hat{y}_t = \circ dW_t - \dfrac{\alpha \hat{y}_t}{2} dt \\ \circ d\hat{z}_t = \dfrac{-\hat{y}_t}{2} \circ dB_t + \dfrac{\hat{x}_t}{2} \circ dW_t - \alpha \hat{z}_t dt \end{cases} \tag{5.4.8}$$

This system is clearly satisfied by the announced functions. $\square$

With these explicit formulae, we can compute the action of right-invariant vector fields on the flow $\hat{G}$.

Proposition 5.4.3. *The right-invariant vector fields X^R, Y^R and Z^R have the following action on $\hat{G}$: for all $g \in \mathcal{H}$ and $t \geq 0$*

$$X^R.\hat{G}_t(g) = e^{-\frac{\alpha t}{2}} X^R(\hat{G}_t(g)),$$
$$Y^R.\hat{G}_t(g) = e^{-\frac{\alpha t}{2}} Y^R(\hat{G}_t(g)),$$
$$Z^R.\hat{G}_t(g) = e^{-\alpha t} Z^R(\hat{G}_t(g)).$$

Proof. We present the calculation for the first vector field only. The two others are similar. For any $g = (x_0, y_0, z_0) \in \mathcal{H}$ and $t \geq 0$, the path given by $a \mapsto (a, 0, 0) * g$ is starting from g with initial speed $X^R(g)$. Then, we have :

$$X^R.\hat{G}_t(g) = \left.\frac{d}{da}\right|_{a=0} \hat{G}_t((a, 0, 0) * g). \tag{5.4.9}$$

Using the formulae of Lemma 5.4.2, we have :

$$\begin{cases} \hat{x}_t((a,0,0) * g) = e^{-\frac{\alpha t}{2}} \left(x_0 + a + \int_0^t e^{\frac{\alpha s}{2}} dB_s \right) \\ \hat{y}_t((a,0,0) * g) = e^{-\frac{\alpha t}{2}} \left(y_0 + \int_0^t e^{\frac{\alpha s}{2}} dW_s \right) \\ \hat{z}_t((a,0,0) * g) = e^{-\alpha t} \left(z_0 + \frac{1}{2} a y_0 + \int_0^t e^{\alpha s} dM_t^{(a,0,0)*g} \right) \end{cases} \tag{5.4.10}$$

and

$$dM_t^{(a,0,0)*g} = \frac{1}{2} e^{-\frac{\alpha t}{2}} \left[\left(x_0 + a + \int_0^t e^{\frac{\alpha s}{2}} dB_s \right) dW_t + \left(y_0 + \int_0^t e^{\frac{\alpha s}{2}} dW_s \right) dB_t \right].$$

So,

$$\begin{aligned} X^R.\hat{G}_t(g) &= e^{-\frac{\alpha t}{2}} \partial_x + e^{-\alpha t} \left(\frac{y_0}{2} + \int_0^t e^{\alpha s} \frac{1}{2} e^{-\frac{\alpha s}{2}} dW_s \right) \partial_z \\ &= e^{-\frac{\alpha t}{2}} \partial_x + e^{-\alpha t} \left(\frac{y_0}{2} + \frac{1}{2} \int_0^t e^{\frac{\alpha s}{2}} dW_s \right) \partial_z \\ &= e^{-\frac{\alpha t}{2}} \left[\partial_x + \frac{1}{2} e^{-\frac{\alpha t}{2}} \left(y_0 + \int_0^t e^{\frac{\alpha s}{2}} dW_s \right) \partial_z \right] \\ &= e^{-\frac{\alpha t}{2}} \left(\partial_x + \frac{1}{2} \hat{y}_t \partial_z \right) \\ &= e^{-\frac{\alpha t}{2}} X^R(\hat{G}(g)_t) \end{aligned}$$

□

Corollary 5.4.4. *The semi-group $\hat{\mathbf{P}}_t$ preserve the monotonicity of $\mathcal{C}_b^1(\mathcal{H})$ functions along each vector fields of the frame (X^R, Y^R, Z^R).*

Proof. Let f be an increasing function along X^R. Then, we have :

$$\begin{aligned}\left\langle \hat{\mathbf{P}}_t f, X^R(g) \right\rangle &= \mathbb{E}\left[\langle df, X^R.\hat{G}_t(g)\rangle\right] \\ &= e^{-\frac{\alpha t}{2}}\mathbb{E}\left[\langle df, X^R(\hat{G}_t(g))\rangle\right] \\ &> 0\end{aligned}$$

The same argument works for Y^R and Z^R. $\square$

Remark that this result is weaker than Proposition 5.3.3 because we do not have the preservation for any right-invariant vector field. We can extend the result to vector fields in $\mathrm{Span}(X^R, Y^R)$. In the Euclidean case, the behaviour of $W_t v$ was determined by the diagonal of $\nabla^2 V$. Here, it appears to be the diagonal values of $\alpha\nabla D$. We will be clarified in the following section this remark. Which choice of connection does make sense? Why does the Ricci term seem to disappear? This proof involves many computational details. We will see a more general and less technical in the next part. Yet, these calculations can be useful to give stochastic proofs of other properties of the generator $L - \alpha D$. We begin with a commutation formula between $\mathbf{P}$ and $\hat{\mathbf{P}}$.

Theorem 5.4.5. *For all $f \in \mathcal{C}_b^0(\mathcal{H})$, for all $t \geq 0$ and $g \in \mathcal{H}$, we have :*

$$\hat{\mathbf{P}}_t f(g) = \mathbf{P}_{\frac{1-e^{-\alpha t}}{\alpha}} f(\delta_{e^{-\frac{\alpha t}{2}}} g).$$

Proof. It is enough to show that for all fixed $t \geq 0$ and for all $g \in \mathcal{H}$, $\hat{G}_t(g)$ and $G_{\frac{1-e^{-\alpha t}}{\alpha}}(\delta_{e^{-\frac{\alpha t}{2}}} g)$ have the same law. We denote $(x_t(g), y_t(g), z_t(g))$ the coordinates of $\hat{G}_t(g)$. Let us fix $t \geq 0$ and $g = (x_0, y_0, z_0) \in \mathcal{H}$. We have :

$$\begin{cases} x_t(g) = x_0 + B_t \\ y_t(g) = y_0 + W_t \\ z_t(g) = z_0 + \dfrac{1}{2}\displaystyle\int_0^t x_s(g) dW_s - \dfrac{1}{2}\int_0^t y_s(g) dB_s \end{cases}. \tag{5.4.11}$$

According to the formulae, the coordinates of both processes are clearly Gaussian, so they are characterized by their two first moments. We have :

$$\begin{aligned}\mathbb{E}\left[x_{\frac{1-e^{\alpha t}}{\alpha}}(\delta_{e^{-\frac{\alpha t}{2}}} g)\right] &= e^{-\frac{\alpha t}{2}} x_0 = \mathbb{E}[\hat{x}_t(g)] \\ \mathbb{E}\left[\left(x_{\frac{1-e^{\alpha t}}{\alpha}}(\delta_{e^{\frac{\alpha t}{2}}} g)\right)^2\right] &= \frac{1-e^{\alpha t}}{\alpha} = \mathbb{E}[(\hat{x}_t(g))^2]\end{aligned}$$

So $x_{\frac{1-e^{\alpha t}}{\alpha}}(\delta_{e^{-\frac{\alpha t}{2}}} g) \sim \hat{x}_t(g)$. The same calculation prove the result for the second coordinate. For the third coordinate, we have :

$$\mathbb{E}\left[z_{\frac{1-e^{\alpha t}}{\alpha}}(\delta_{e^{-\frac{\alpha t}{2}}} g)\right] = e^{-\alpha t} z_0 = \mathbb{E}[\hat{z}_t(g)]$$

$$\mathbb{E}\left[(z_{\frac{1-e^{\alpha t}}{\alpha}}(\delta_{e^{-\frac{\alpha t}{2}}} g))^2\right] = e^{-\frac{\alpha t}{2}} \frac{1-e^{\alpha t}}{4\alpha}(x_0^2+y_0^2) + \frac{(1-e^{-\alpha t})^2}{4\alpha^2} = \mathbb{E}\left[(\hat{z}_t(g))^2\right]$$

So we have : $z_{\frac{1-e^{\alpha t}}{\alpha}}(\delta_{e^{-\frac{\alpha t}{2}}} g) \sim \hat{z}_t(g)$. This ends the proof. □

This commutation formula can be seen as a generalization of Mehler formula (see [29]). It brings another proof of corollary 5.4.4, using the result for $\mathbf{P}$.

As D is not a gradient, $\hat{\mathbf{P}}$ is not reversible. Nevertheless, it admit an invariant measure. We denote by K_t the density of $G_t(e)$. The formula of Theorem 5.4.5 suggest a link between both the invariant measure of $\hat{\mathbf{P}}$ and K.

Corollary 5.4.6. *The invariant law of $\hat{\mathbf{P}}$ is $k_{1/\alpha}(x,y,z)dxdydz$ i.e. :*

$$\int_{\mathcal{H}} (L-\alpha D) f K_{1/\alpha} dxdydz = 0, \ \forall f \in \mathcal{C}_c^\infty(\mathcal{H}).$$

Proof. The Brownian motion has a property of invariance in law by dilation : for all $t \geq 0$ and $c > 0$ we have $\delta_{\frac{1}{\sqrt{c}}} G_{ct}(e) \sim G_t(e)$. Then

$$\mathbb{E}\left[f(\delta_{\frac{1}{\sqrt{c}}} G_{ct}(e))\right] = \mathbb{E}\left[f(G_t(e))\right]$$

$$\int_{\mathcal{H}} f(\frac{x}{\sqrt{c}}, \frac{y}{\sqrt{c}}, \frac{z}{\sqrt{c}}) K_{ct}(x,y,z) dxdydz = \int_{\mathcal{H}} f(x,y,z) K_t(x,y,z) dxdydz$$

$$\int_{\mathcal{H}} f(x,y,z) c^2 K_{ct}(\sqrt{c}x, \sqrt{c}y, \sqrt{c}z) dxdydz = \int_{\mathcal{H}} f(x,y,z) K_t(x,y,z) dxdydz$$

So for all $t \geq 0$ and $(x,y,z) \in \mathcal{H}$, we have :

$$t^2 K_{t/\alpha}(\sqrt{t}x, \sqrt{t}y, \sqrt{t}z) = K_{1/\alpha}(x,y,z).$$

By differentiating with respect to t and evaluating in $t = 1$, we have :

$$(2id + \frac{1}{\alpha} L + D) K_{1/\alpha} = 0. \tag{5.4.12}$$

On the other hand, integration by parts, for all $f, g \in \mathcal{C}_c^\infty(\mathcal{H})$, we have :

$$\int_{\mathcal{H}} (L-\alpha D) f g \, d\,\mathrm{vol} = \int_{\mathcal{H}} f (L + \alpha D + 2\alpha\,\mathrm{id}) g \, d\,\mathrm{vol} \tag{5.4.13}$$

This ends the proof. □

5.5 Deformed parallel translation on Lie groups

In this section, we study semi-groups of a general diffusion on a Lie group G, with generator $L = \frac{1}{2}\Delta + U$. The goal of this section is to obtain a condition on the potential U for preserving the monotonicity of function along a right-invariant frame. For that, we defined a deformed parallel translation and we establish a intertwining. Finally, we illustrate the result with some family of examples in the Heisenberg group.

We endow G with its left-connection ∇^L. By Δ we refer to the associated Laplacian. Let U be a smooth vector field, not necessarily left-invariant, nor a gradient field. We consider the flow defined by the following Ito equation:

$$\begin{cases} d^{\nabla^L} X_t(g) = dL_{X_t} dB_t + U(X_t)dt \\ \qquad X_0(g) = g \end{cases} \tag{5.5.1}$$

where B_t is a Brownian motion in $\mathfrak{G}$. This defines a flow of the diffusion with generator $L = \frac{1}{2}\Delta + U$. We denote by $\mathbf{P}$ the associated semi-group. As in the euclidean case, we want to study the spatial derivative of this flow :

$$W_t^g = dX_t(g) : T_g G \to T_{X_t(g)}G. \tag{5.5.2}$$

This W^g plays the role of deformed parallel translation in $\mathbb{R}^n$. Here we have chosen the definition such that the processes $X(g)$ and W^g are intertwined by construction. We will fulfil the comparison with the classical deformed parallel transform. First, we establish a covariant derivative equation satisfied by W^g. In the case of (3.2.4), the choice of the connection to define the covariante derivative is obvious. Here, as we do not use the Riemannian connection, the choice is less natural. We want to study the derivative of $\mathbf{P}_t f$ with respect to right-invariant vector fields. That is why we chose the right-connection, ∇^R, defined in (2.2.2). For this connection, the parallel translation along a curve c is the right-multiplication differential:

$$/\!/_t^R = d_{c(0)} R_{c(0)^{-1}c(t)}. \tag{5.5.3}$$

Theorem 5.5.1. *With the previous notations, we have :*

$$D^R W_t^g v = \nabla^L_{W_t^g v} U(X_t) dt, \tag{5.5.4}$$

where $D^R = /\!/_t^R d(/\!/_t^{R^{-1}} \cdot)$.

Proof. Let $c(\cdot, a)_a$ be a smooth family of smooth curves in G. By definition of torsion and with the commutation property 2.2.1, we have :

$$\nabla^R_{\partial_t c}\partial_a c = \nabla^R_{\partial_a c}\partial_t c + [\partial_a c, \partial_t c] + T^R(\partial_t c, \partial_a c)$$

$$
\begin{aligned}
&= \nabla^R_{\partial_a c}\partial_t c + T^R(\partial_t c, \partial_a c) \\
&= \nabla^L_{\partial_a c}\partial_t c.
\end{aligned}
$$

Then, with Stratonovich transfer principle, this equality is satisfied by the family of semi-martingales $Y(t,a) = X_t(g\exp(av))$. By evaluating at $a = 0$, we obtain :

$$D^{SR} W_t^g v = \nabla^L_{W_t^g v} \circ dX_t(g), \tag{5.5.5}$$

where D^{SR} denote the Stratonovich covariant derivative. Now, we need to change from Stratonovich derivative to Ito one. As the connection is left-invariant, the correction terms $\frac{1}{2}\nabla^L_{H_i^L} H_i^L$ vanish. So, for all $g \in G$, the process $X(g)$ satisfies :

$$\circ dX_t(g) = \sum H_i^L(X_t(g)) \circ dB_t^i + U(X_t(g))dt. \tag{5.5.6}$$

Then, the martingale part of $D^{SR} W_t^g v$ is

$$\sum_i \nabla^L_{W_t^g v} H_i^L(X_t(g)) dB_t^i. \tag{5.5.7}$$

So $D^{SR} W_t^g(v)$ has no martingale part. Then it is equal to the Ito covariant derivative $D^R W_t^g(v)$:

$$D^R W_t^g(v) = \nabla^L_{W_t^g(v)} U(X_t(g)).$$

$\square$

The obvious advantage of the right-covariant derivative is the vanishing of the martingale part. If we calculate the left-covariant derivative, the we would not have finite variation processes.

As in the classical case, the definition of a semi-group on 1-form associated to W requires an additional assumption of boundedness, a kind of Bakry-Émery criterion. The problem does not come from the lack of symmetry of $\nabla^L U$, Bakry-Émeri criterion is adaptable to such potentials, but the problem comes from $/\!/^R$ which is not isometric : we have

$$d|W_t^g v|^2 \neq 2\langle W_t^g v, D^R W_t^g v\rangle.$$

A way to bypass this issue would be to look at the left-covariant derivative $D^L W_t^g$ but then, we must deal with a martingale part as in Proposition 4.3.2. By comparison, we conjecture that the boundedness of the bilinear operator $\nabla^L U$ and of some norm of the right-translation should be suitable. We the case of bi-invariant metric, we ha the following result.

Lemma 5.5.2. *Assume that G has a bi-invariant metric and that $\nabla^L U < +\infty$, then we have the following stochastic representation of $d\mathbf{P}$: for all $f \in \mathcal{C}_b^1(G)$, for all $t \geq 0$, for all $g \in G$ and for all $v \in T_g G$:*

$$\langle d\mathbf{P}_t f(g), v\rangle = \mathbb{E}\left[\langle df, W_t^g(v)\rangle\right]. \tag{5.5.8}$$

Proof. We have :

$$\begin{aligned} d|W_t^g v|^2 &= d\langle /\!\!/_t^{R^{-1}} W_t^g v, /\!\!/_t^{R^{-1}} W_t^g v\rangle \\ &= 2\langle W_t^g v, D^R W_t^g v\rangle \\ &= 2\langle W_t^g v, \nabla^L U W_t^g v\rangle dt \\ &\leq 2k|W_t^g v|^2 dt \end{aligned}$$

The proof ends as in Proposition 3.4.3. □

We denote by α_i the dual basis of T^*M defined by : for all i, j, $\langle \alpha_i, \mathbb{H}_j^R\rangle = \delta_{ij}$. As $(H_i)_i$ is not orthonormal, α_i is not the adjoint of H_i^R for the metric. The forms α_i are right-invariant : $D^R\alpha_i = 0$.

Theorem 5.5.3. *Assume that the potential U satisfies : for all $i \neq j$ and all $g \in G$,*

$$\left\langle \alpha_i, \nabla^L_{H_j^R} U \right\rangle \geq 0,$$

then W^g preserve the non-negativity of vector in the basis $(H_i^R)_i$.

Proof. We have

$$\begin{aligned} d\left\langle \alpha_i(X_t), W_t^g(v)\right\rangle &= \left\langle \alpha_i(X_t), D^R W_t^g(v)\right\rangle \\ &= \left\langle \alpha_i(X_t), \nabla^L_{W_t^g(v)} U(X_t(g))\right\rangle dt \\ &= \sum_{j=1}^n \left\langle \alpha_j, W_t^g(v)\right\rangle \left\langle \alpha_i(X_t), \nabla^L_{H_i^R} U(X_t(g))\right\rangle dt \end{aligned}$$

We conclude as in the euclidean case. □

This result is very similar to the Euclidean one in Theorem 5.2.2. Firstly, we prove the preservation of positivity almost surely, stronger than the claim. Secondly, it prove the preservation of non-negativity in a basis but not coordinate by coordinate as in Proposition 5.4.4. Remark that the theorem is satisfied for all right-invariant basis, with the associated condition. Together with the intertwining (5.5.8), the criterion of Theorem 5.5.3 prove that the semi-group **P** preserve the monotonicity of function in the basis $(H_i^R)_i$. Now, we suppose that U is a gradient field, for the right-connection :

$$U = -\nabla^R V = \sum_{i=1}^n \langle dV, H_i^R\rangle H_i^R, \tag{5.5.9}$$

with V a smooth function. We denote by Γ_{ij}^k are the Christoffel symbols of the left-connection in the map associated to the right-invariant vector fields :

$$\nabla^R_{H_i^R} H_j^R = \sum_{k=1}^{n} \Gamma_{ij}^k H_k^R. \tag{5.5.10}$$

With this special form of U, Theorem 5.5.3 reformulate as follow.

Corollary 5.5.4. *If the potential V satisfies : for all $i \neq j$, for all $g \in G$,*

$$\forall i \neq j, \quad \nabla^{R2} V(H_i^R, H_j^R) + \sum_{k=1}^{n} \Gamma_{jk}^i \langle dV, H_k^R \rangle \leq 0,$$

then W^g preserve the non-negativity of vector in the basis $(H_i^R)_i$.

In the Euclidean space, where the metric is flat, the Christoffel symbols vanish. Then, Theorem 5.2.2 can be seen as a corollary.

We finish with examples of semi-groups on the Heisenberg group $\mathcal{H}$. Firstly, Theorem 5.5.3 can explain Proposition 5.4.4. Indeed, the vector field $U = -\alpha D$ satisfies in the right-invariant basis (X^R, Y^R, Z^R) :

$$\nabla^L U = -\frac{\alpha}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}. \tag{5.5.11}$$

Then, all th terms outside the diagonal vanish. Moreover, invariant connections are flat : their Ricci tensor vanish. The equivalent of the potential $\mathcal{M}$ resumes as $\nabla^L U$. This explains the exponential behaviour of the flow's partial derivative in Proposition 5.4.3. Now, we look for some small deformations of D which still satisfy the criterion.

Proposition 5.5.5. *Let's u and v be real functions such that $u \geq 0$ and $v \leq 0$ and γ a positive constant. The the vector field*

$$U = \gamma D + u(x) X^R + v(y) Y^R$$

satisfies the condition of Theorem 5.5.3.
Moreover, if u and v are increasing, then the vector field

$$U = \gamma D + u(z) X^R + v(z) Y^R$$

satisfies this condition too.

In a wider way, we have the following result.

Proposition 5.5.6. *Let's α, β and γ be differentiable functions such that :*

$$\alpha(x) - \frac{x}{2}\gamma'(z) \geq 0 \quad \text{and} \quad \beta(y) - \frac{y}{2}\gamma'(z) \leq 0, \quad \forall (x, y, z) \in \mathcal{H}.$$

Then the vector field $U = \alpha(x) X^R + \beta(y) Y^R + \gamma(z) Z$ satisfy the condition of Theorem 5.5.3.

5.6 Perspectives

At the end of this chapter, there are still many questions to investigate. Firstly, concerning the deformed parallel translation introduced in (5.5.2), we did not finish the comparison with the classical one (defined with the Levi-Civita covariant derivative). We could to investigate its property as diffusion on the tangent bundle TG. We can conjecture that the generator on 1-forms has the shape $\frac{1}{2}\tilde{\Delta} + \nabla^R_U + \nabla^L U$ where $\tilde{\Delta}$ would be an horizontal Laplacian on 1-form for the right-connection. The detail that stops the analogy with the classical case is that right-translations are not isometric. However, we can conjecture that the generator on 1-form of W^g will satisfies an interesting commutation property with the generator L of the diffusion on G. Indeed, a straightforward calculation shows that :

$$\langle d\langle df, U\rangle, v\rangle = \langle \nabla^R_U df, v\rangle + \langle df, \nabla^L_v U\rangle. \tag{5.6.1}$$

A second point is the intertwining at the level of semi-group. This question has already been raised in the previous section. It is not clear which covariant derivative can give a good criterion. In Chapter 4, we did not manage to obtain a criterion guaranteeing both existence and intertwining, because of the martingale part.

A third point is the generalization of the FKG property to Lie groups. In $\mathbb{R}^n$, we need the reversibility and ergodicity of the diffusion to use the covariance representation of Proposition 3.3.1. In our case, it is not so clear that we have these properties, even if U is a right-gradient. We do not even have examples of such a gradient field satisfying the criterion of Corollary 5.5.4.

Finally, in a more prosaic consideration, it is not clear that in our examples, the set of $\mathcal{C}^1_b(G)$ increasing functions is not empty. That is why we still do not have examples of increasing preservation nor FKG property in Lie groups.

Part III

Brenier-Schrödinger problem

"Il n'est pas un moi. Il n'est pas dix moi. Il n'est pas de moi. Moi n'est qu'une position d'équilibre. (Une entre mille autres continuellement possibles et toujours prêtes.) Une moyenne de "moi", un mouvement de foule."
Henri Michaux, *Plume*

Chapter 6

From Euler to Brenier-Schrödinger

This chapter is an introduction to Brenier-Schrödinger problem. We recall some notion of fluid evolution equations, the genesis of Brenier-Schrödinger problem, known results and we present the different avenues we will explore in the following chapters. The presentation of the problem is mostly inspired by those from [5].

Contents

6.1 Introduction

In this part, we study various questions about Brenier-Schrödinger problem in a compact Riemannian manifold with boundary. The goal of this chapter is to introduce the problem and its background. This problem is a Lagrangian, or principle of least action, approach of Navier-Stokes equation. This approach has been develop for Euler equation by Brenier in [21] in the continuity of Arnold geodesic problem [7], with a relevant back and forth between solutions of fluid equation and minimising problem. Euler equation describe inviscid fluid. The usual description of viscous fluid is given by Navier-Stokes equation (6.5.1). The viscosity is modelled by a Laplacian term. Thinking about stochastic calculus, Laplacian strongly suggests the introduction of Brownian path measures and the substitution of kinetic energy in Brenier problem by a stochastic kinetic energy. This generalization is introduced in [1]. Since its initial formulation, the problem has evolved. In particular, the stochastic kinetic energy has been be related to a relative entropy with respect to the Brownian law. It leads to the actual formulation (BS) and the denomination as Brenier-Schrödinger or Brödinger, or Bredinger in the literature, as a mix between Brenier energy minimisation problem and Schrödinger entropy minimisation problem. The results of this part come from a work with David Garcia Zelada (see [42]). Following the approach of [5], we extend its results on kinetic of solution and criterion of existence to compact manifold with boundary. We bring improvement for the impermeability question and in examples of problems admitting solutions.

Let us summarise this chapter. In Section 6.2, we introduce our state space and the notions of path measures. In Section 6.3, we recall some result on fluid evolution and Euler equation. The Section 6.4 explains the logic from evolution equation to minimisation problem. It motivates the Lagrangian approach of Navier-Stokes equation. In section 6.5, we introduce Brenier-Schrödinger problem and we expose the topics we will develop in the following chapters.

6.2 Notations

In this section, we fix some notation specific to Part III. We define the path space, marginal measures and the relative entropy.

Let $(M, \langle\cdot,\cdot\rangle)$ be a smooth compact Riemannian manifold with boundary, of dimension n. It is our state space. We denote by ∂M its boundary and $\mathring{M}$ its interior. We denote by Ω the path space $\mathcal{C}^0([0,1], M)$. It is the space of motions of a particle in M. The set of Borel probability measures of Ω is denoted by $\mathcal{P}(\Omega)$. The canonical process X on Ω is defined as

$$X_t(\omega) = \omega_t,\ \forall t \in [0,1],\ \forall \omega \in \Omega.$$

The canonical process generates the canonical filtration $(\mathcal{F}_t)_{t\in[0,1]}$ defined by :

$$\mathcal{F}_t = \sigma(X_{[0,t]}),\ \forall t \in [0,1].$$

For a probability $Q \in \mathcal{P}(\Omega)$ and a time $0 \leq t \leq 1$, we denote by Q_t the marginal measure $X_{t\#}Q$. It is the law of the random variable X_t under Q. It represents the density of particles at time t. In the same way, the probability $Q_{01} \in P(M^2)$ is law of the couple (X_0, X_1). It represents the endpoints distribution of particles.

A probability measure $Q \in \mathcal{P}(\Omega)$ can be disintegrated with respect to the initial position :

$$Q = \int_M Q^x(\cdot) Q_0(dx),$$

where $x \mapsto Q^x = Q(\cdot \mid X_0 = x) \in \mathcal{P}(\Omega)$ is measurable. Similarly, the disintegration with respect to the endpoints position is meaningful and it defines the bridge measure $Q^{xy} = Q(\cdot \mid X = 0 = x, X_1 = y) \in \mathcal{P}(\Omega)$ for $x, y \in M$.

For P and R in $\mathcal{P}(\Omega)$, the relative entropy of P with respect to R is defined as :

$$H(P|R) = \int_\Omega \log \frac{dP}{dR}\, dP \in]-\infty, +\infty], \tag{6.2.1}$$

whenever this expression is meaningful. This entropy has an interpretation in convex optimisation which can be found in [55] and recall in Chapter 7. It satisfies an useful additive decomposition formula :

$$H(P|R) = H(f_\# P | f_\# R) + \int_{\mathcal{Z}} H(P^{f=z} | R^{f=z}) f_\# P(dz), \tag{6.2.2}$$

where $\mathcal{Z}$ is a Polish space (often M or M^2), $f : \Omega \to \mathcal{Z}$ measurable (often X_0 or (X_0, X_1)) and $P^{f=z}$ a version of the conditional probability $P(\cdot \mid f = z)$. See [44] for details on this formula. This formula will be a key argument in Chapter 9 to obtain finite entropy condition.

6.3 Euler equation

In this section, we introduce the notions of fluid evolution as Eulerian description, incompressibility and impermeability. The goal of this section is to present Euler equation.

There are two different descriptions of a fluid evolution : the Lagrangian and the Eulerian. The Lagrangian coordinates are given by a function

$$q : [0,1] \times M \to M,$$

giving the position at a given time of a particle starting from a given initial position. The Lagrangian description is a particle-wise approach. The Eulerian description of the flow is a vector field

$$v : (t,x) \in [0,1] \times M \mapsto v(t,x) \in T_x M,$$

giving the velocity of the fluid at a given time and a given position. As they describe the same phenomenon with different point of view, there is a link between these two descriptions. From Lagrangian to Eulerian, v is the velocity of the flow q.

Equivalently, from Eulerian to Lagrangian, q is the integral curve of the vector field v. We have : $\partial_t q(t,x) = v(t, q(t,x))$, for all $(t,x) \in [0,1] \times M$. Remark that this link between the two descriptions makes sense if the trajectories of particles do not cross, i.e if for all t, $x \mapsto q(t,x)$ is injective.

After position and velocity, the quantity which characterize the dynamic is the acceleration. In Lagrangian coordinates, the acceleration of a particle is just the second derivative $\partial_t^2 q$. Expressed in Eulerian coordinates, we have :

$$\partial_t^2 q(t,x) = \partial_t v(t, q(t,x)) + \nabla_v v(t, q(t,x)),\ \forall (t,x) \in [0,1] \times M.$$

The operator $D_t = \partial_t + \nabla_v$ is called the convective derivative.

Let us introduce two important notions in the description of fluid evolution. A fluid in M is said incompressible if its flow preserves the volume. As a condition on the vector field v rather than on the flow, it is equivalent to

$$\mathrm{div}(v) = 0,\ \forall (t,x) \in [0,1] \times M. \tag{6.3.1}$$

The incompressibility is a particular case of the mass preservation, or continuity equation, for non-homogeneous fluid :

$$\partial_t \mu_t + \mathrm{div}(\mu_t v) = 0,$$

where μ_t is the density of particle at time $t \in [0,1]$. It states that there is not any creation nor annihilation of particles.

We denote by ν the normal inner vector field at ∂M. The impermeability condition of the fluid is

$$\langle v, \nu \rangle = 0,\ \forall x \in \partial M. \tag{6.3.2}$$

In other word, the velocity v is tangent to the boundary. This means that there is not any particle entering or leaving M.

The Euler equation for incompressible fluid with impermeability condition is the system

$$\begin{cases} \partial_t v + \nabla_v v + \nabla p = 0, & (t,x) \in [0,1] \times M \\ \mathrm{div}(v) = 0, & (t,x) \in [0,1] \times M \\ \langle v, \nu \rangle = 0, & (t,x) \in [0,1] \times \partial M \\ v(0,\cdot) = v_0, & x \in M \end{cases} \tag{6.3.3}$$

where v_0 is a given initial velocity field. The scalar pressure field $p : [0,1] \times M \to \mathbb{R}$ is unknown. A solution of Euler equation is a couple (v,p). The first equation of this system,

$$D_t v = -\nabla p,$$

can be interpreted as Newton's second law with a force $-\nabla p$, derived from a scalar potential.

6.4 From Euler to Brenier

In this section, we present the steps from Euler evolution equation to Brenier minimising problem. It motivates our approach of Navier-Stokes equation via Brenier-Schrödinger problem.

The first step from Euler equation to Brenier problem is due to Arnpold. In [7] he proposed an analytical mechanics approach of Euler equation. Rather than a the differential system satisfied by the velocity, he looked to a variational problem satisfied by the trajectory. According to the principle of least action, trajectories minimise a functional. We denote by G_{vol} the group of volume and orientation preserving diffeomorphisms of M. The variational problem is :

$$\int_{[0,1]\times M} |\partial_t q_t(x)|^2 \, dtdx \to \min; [q_t \in G_{\text{vol}},\ \forall 0 \leq t \leq 1], q_0 = \text{id}, q_1 = h, \tag{6.4.1}$$

for a prescribe endpoint $h \in G_{\text{vol}}$. Then if $(q_t)_{0\leq t\leq 1}$ is a minimiser, the vector field $v(t,z) = \partial_t q_t(q_t^{-1}(z))$ is a solution of a endpoint variation from system (6.3.3) : rather than an initial condition v_0, the integral curves must transport x to $h(x)$ in time 1. Arnold proves in [7] that the solution of the minimising problem are geodesics for a right-invariant metric on G_{vol} between the two element of G_{vol} : the identity id and h. Th geodesic problem (6.4.1) have been studied in [31] for h a small perturbation of id and in [73] and [74] for examples where such geodesics do not exist.

The final step is reached by Brenier in [21]. He introduced a relaxation of problem (6.4.1) for compact domain $\mathcal{X}$ in $\mathbb{R}^n$: rather than looking for trajectories, he looked at measure on trajectories, minimising an average action, satisfying a marginal incompressibility constraint and a endpoints constraint. Formally, the Brenier problem is :

$$\mathbb{E}_Q\left[\int_0^1 |\dot{X}_t|^2 \, dt\right] \to \min; Q \in \mathcal{P}(\Omega), [Q_t = \text{vol}, \forall 0 \leq t \leq 1], Q_{0,1} = \pi, \tag{6.4.2}$$

where $\pi \in \mathcal{P}(\chi^2)$ and the process $\dot{X}$ is defined on absolutely continuous path $\omega \in \Omega$ by $\dot{X}_t(\omega) = \dot{\omega}_t$. In the above functional, the integral is understood as $+\infty$ if ω is not absolutely continuous. Then, a solution of problem (6.4.2) has support on absolutely continuous path only. The marginal constraint $Q_t = \text{vol}$ plays the role of volume preservation in problem (6.4.1). The endpoints constraint $Q_{01} = \pi$ is a relaxation of $Q_{01}(dxdy) = \text{vol}(dx)\delta_{h(x)}(dy)$ in problem (6.4.1). Brenier proved in [21] that a path measure P satisfying

$$\begin{cases} \ddot{X}_t + \nabla p(t, X_t) = 0, \ \forall t, P\text{-a.s} \\ P_t = \text{vol} \ \forall t \quad \text{and} \quad P_{01} = \pi \end{cases}$$

is a solution of (6.4.2). Conversely, if P is a solution of (6.4.2), the velocity measure σ on $[0,1] \times \mathcal{X} \times R^n$ defined by

$$\int_{[0,1]\times\mathcal{X}\times R^n} f(t,x,v)\,\sigma(dtdxdv) = \mathbb{E}_P\left[\int_0^1 f(t,X_t,\dot{X}_t)\,dt\right], \tag{6.4.3}$$

is a solution of Euler equation in a certain sense (see [22] for details). Then we somehow have an equivalence between the Euler equation, a notoriously difficult Cauchy problem, and the Brenier problem, a minimising problem. This is the angle of attack we develop for Navier-Stokes equation.

6.5 Brenier-Schrödinger problem

In this section, we introduce the Brenier-Schrödinger problem. We present Navier-Stokes equation and give the heuristic which leads to Brenier-Schrödinger problem.

While Euler equation describes the evolution of non-viscous fluid, Navier-Stokes equation takes into account a viscosity term :

$$\begin{cases} \partial_t v + \nabla_v v - a\Box v + \nabla p = 0, & (t,x) \in [0,1]\times M \\ \mathrm{div}(v) = 0, & (t,x) \in [0,1]\times M \\ \langle v, \nu\rangle = 0, & (t,x) \in [0,1]\times \partial M \\ v(0,\cdot) = v_0, & x \in M \end{cases} \tag{6.5.1}$$

where $\Box$ is the extension of Hodge-de Rham Laplacian defined in (2.5.1) to vector fields and $a > 0$. The term $a\Box$ represent a viscosity force. Remark that it is a different choice from [4] where the the Laplacian used is $\hat{\Box} = \Box - 2\,\mathrm{Ric}$ but in flat spaces where Ric vanishes, as torus $\mathbb{T}^n$, both are equal. This problem is notoriously difficult and it is not the goal of our work to solve Navier-Stokes equation. We want to study a relaxed problem, even where classical solutions do not exist, in the style of Brenier problem.

The first generalisation of Brenier problem to viscous fluid comes from [1]. The problem is formulated as the minimisation of a stochastic kinetic energy on law of Brownian motion with drift. In this problem, the drift is used as a generalisation of the classical velocity $\dot{X}_t$. Since this founding article, other authors took on the subject and brought the problem to its actual entropy formulation. Remarks that Yasue proposed a variational approach of Navier-Stokes equation in [77]. We find in this article all the main tools, from the Nelson derivative to the entropy formula.

Let $\sigma : M \times \mathbb{R}^n \to TM$ such that $\sigma(x) \in L(\mathbb{R}^m, T_xM)$, $\sigma\sigma^*(x) = \mathrm{id}_{T_xM}$. Such a σ exists when M is embedded in $\mathbb{R}^m$, for example.

Definition 6.5.1. *A stochastic process X is a reflected Brownian motion on M if it solves the Skorokhod problem:*

$$dX_t = \sigma(X_t)dW_t + \nu_{X_t}dL_t, \tag{6.5.2}$$

with W is a Brownian motion in $\mathbb{R}^m$ and L is a non-decreasing process such that

$$\int_0^1 \mathbb{1}_{\mathring{M}}(X_s)dL_s = 0.$$

The process L is the local time of the reflected Brownian motion at ∂M. We denote by R^x the law of the reflected Brownian motion on M starting from x and R the law of the reflected Brownian motion starting from the uniform distribution on M, vol. The measure R is the reversible law of the reflected Brownian motion on M. These laws are well and uniquely defined (see [49] and [6]).

The Brenier-Schrödinger problem, or Brödinger problem, is a entropy minimisation problem. Let $\mathcal{T}$ a measurable subset of $[0, 1]$, $(\mu_t)_{t\in\mathcal{T}}$ a set of measures in $\mathcal{P}(M)$ indexed by $\mathcal{T}$, and $\pi \in \mathcal{P}(M^2)$. The Brenier-Schrödinger problem is :

$$H(Q|R) \to \min,\ Q \in \mathcal{P}(\Omega),\ [Q_t = \mu_t,\ \forall t \in \mathcal{T}],\ Q_{01} = \pi. \tag{BS}$$

This problem is a mix between Brenier problem, minimization of energy under marginal and endpoints constraints, and Schrödinger problem, minimisation of entropy under marginal constraints (see [41]). As explained in [5], the entropy with respect to Brownian law is linked to a stochastic kinetic energy of [1]. In Chapter 7, we prove that a solution is a semi-martingale measure. We will defined this stochastic kinetic energy and give a prove this link.

This problem is a strictly convex problem with linear constraints. There is a general result of existence and uniqueness from [16], for more general state space M and reference measure R. In our case, it is the following.

Theorem 6.5.2 ([16])**.** *The problem (BS) admits a solution if and only if there exists $Q \in \mathcal{P}(\Omega)$ such that $Q_t = \mu_t$ for all $t \in \mathcal{T}$, $Q_{01} = \pi$ and $H(Q|R) < +\infty$. In this case, the solution is unique.*

This theorem does not give easily verifiable criterion of existence. In [5], there is a result when the state space is a torus $\mathbb{T}^n$, the reference measure is the reversible Brownian law and for the incompressible marginal constraint $\mu_t = \text{vol}$ for all $t \in [0, 1]$. It is proved that there exists solutions if and only if $H(\pi|R_{01}) < +\infty$. This part of the problem will be studied in Chapter 9. We prove a similar criterion for compact manifold on which the isometries acts transitively. We give a method to find examples in manifold with boundary (or with corner) by quotient. We also explore a non incompressible problem in a non-compact space : Gaussian measures in $\mathbb{R}^n$.

As we already said, Brenier-Schrödinger problem does not resolve Navier-Stokes equation. Yet, there is some results of links between both problem similar to the links between Euler equation and Brenier problem. From Navier-Stokes to Brenier-Schrödinger, we can cite this theorem, rewritten with our formalism.

Theorem 6.5.3 ([4]). *Let v be a regular solution on $[0,T]$ of the incompressible Navier-Stokes equation on $\mathbb{T}^n$:*

$$\partial_t v(t,x) + \nabla_v v(t,x) - a\Box v(t,x) + \nabla p(T-t,x) = 0$$

for a regular pressure p such that $\nabla^2 p(t,x) \leq \frac{\pi^2}{T^2}\,\mathrm{id}$. Consider the process g defined by

$$dg_t = \sqrt{2a}\,dB_t - v(T-t,g_t)dt$$

with B a Brownian motion and $g_0 = \mathrm{vol}$. Then the law Q of g minimise $H(\cdot|R)$ in the class of laws of process

$$\left\{\tilde{g}; d\tilde{g}_t = \sqrt{2a}\,dB_t - u(t,\tilde{g}_t)dt, \tilde{g}_0 = g_0, \tilde{g}_T = g_T, u \text{ regular}\right\}.$$

The reciprocal result, from Brenier-Schrödinger to Navier-Stokes, is the topic of Chapter 8. In [5], it is proved, in the cases of $M = \mathbb{T}^n$ and $M = \mathbb{R}^n$, that if P is a regular solution of (BS), notion to be cleared, its final point conditional stochastic backward velocity satisfies the Newton part of Navier-Stokes equation and its current velocity satisfies the continuity equation. We will defined this notions and extend these results to a compact manifold with boundary. Particularly, we are interested in the impermeability condition which is leaved out in toruses and Euclidean spaces.

Chapter 7

Shape of a general solution

We present the characteristic of a general solution of problem BS. The main result is the semi-martingale characterisation from a Girsanov theorem under finite entropy.

Contents

7.1 Introduction

In this chapter, we are interested in the shape of solutions to BS problem, when they exist. We a looking to structural characterisation of solution. For example, in the case of Brenier problem (6.4.2), from the very nature of the problem, a solution is a measure on absolutely continuous functions. On one hand, a solution of our problem is a measure dominated by a semi-martingale measure. Then, Girsanov theorem prove that a solution is a semi-martingale measure. Te drift, or finite variation part, is a parameter wich characterise the solution. Besides of being absolutely continuous with respect to R, a solution has finite relative entropy. In the Euclidean space, this additional condition allows Léonard in [55] to improve significantly Girsanov theorem : boundedness property of the drift, expressions of density and entropy. On the other hand, the reference measure R has also Markovian properties. The transmission of those properties to solutions has been studied in [16]. A solution will not be Markovian but reciprocal. This gives a new characterisation with other parameters. Then, BS problem can be treated as a convex optimisation problem as in [5]. From the study of primal and dual problems arises a special form on the previous parameters which leads to the notion of regular solution.

Let us summarize this chapter. In Section 7.2, we present the Girsanov theory with Léonard point of view adapted to manifolds. The main results will be applied in Section 7.3 to describe the semi-martingale property of a solution. This will enlighten the link between Brenier problem and Brenier-Schrödinger problem. We finish, in Section 7.4 with the reciprocal measure point of view of solutions. This last section will not contain original work. It makes a link with the study of kinematic.

7.2 Girsanov theorem

In this section, we establish a Girsanov theorem, for measure of finite entropy with respect to a semi-martingale, the existence of a velocity vector field and an explicit expression of the density. This result and its proof are generalizations of [55] to a manifold setting (with or without boundary). This section is mostly translation from the Euclidean language to the manifold one. It seems interesting to fully develop it here for two reasons. Firstly, it gives a nice view of Girsanov theory and deserves to be written in the manifold setting. It is also an occasion to recall some classical but useful result on semi-martingale. Secondly, the three main results presented here will enlighten properties of the solution of BS in Section 7.3.

The Léonard approach of Girsanov theory is based on the variational view of the entropy, recalled in the following lemma.

Lemma 7.2.1 (variational representations of the relative entropy [55]). *Let Q be probability measure on some space Ω.*

1. *For any probability P on Ω, we have :*

$$\sup\left\{\mathbb{E}_P[u] - \log(\mathbb{E}_Q[e^u]) : u \in L^\infty(P)\right\} = \begin{cases} H(P|Q) \in [0, +\infty], & \textit{if} \quad P \ll Q \\ +\infty, & \textit{otherwise} \end{cases}$$

2. *In addition, if $H(P|Q) < +\infty$, then for all measurable functions u such that $e^u \in L^1(Q)$, then $u \in L^1(P)$ and we have :*

$$H(P|Q) = \sup\left\{\mathbb{E}_P[u] - \log(\mathbb{E}_Q[e^u]) : e^u \in L^1(Q)\right\}.$$

Let M be a smooth complete Riemannian manifold, with or without boundary. Let Ω be the paths space of continuous functions $\omega : [0, 1] \to M$ and X the associated canonical process on Ω i.e for all $\omega \in \Omega$, and $t \in [0,1]$, $X_t(\omega) = \omega_t$. We denote by $(\mathcal{F}_t)_{t\geq 0}$ the canonic filtration. We complete this filtered space with a reference probability measure R solving the martingale problem $\mathcal{MP}(B,A)$, where B denotes a drift with bounded variations and A a quadratic variation operator. This means that R is defined as a probability measure such that, for all $f \in \mathcal{C}_c^\infty(M)$, the process

$$M_t^f := f(X_t) - f(X_0) - B_t(df) - \frac{1}{2}A_t(\nabla^2 f),\ \forall t \in [0,1], \tag{7.2.1}$$

is a R local martingale. Path-wise, a drift B has to be understood as an integral operator on 1-forms valued process :

$$B_t(\alpha,\omega) = \int_0^t \langle \alpha_s(\omega_s), U_s(\omega)\rangle\, dg_s(\omega), \tag{7.2.2}$$

where $\alpha_t \in \Gamma(T^*M)$ for all t, U is a TM-valued adapted process above X (such that $U_s(\omega) \in T_{\omega_s}M$) and $g : [0,1]\times\Omega \to \mathbb{R}$ is path-wise a function with bounded variations. The notation $dB = Udg$ will be used as an infinitesimal vector field along X. The main examples of functions g are : $g_t(\omega) = t$ and $g_t(\omega) = L_t(\omega)$ the local time when M has boundaries. Similarly, the quadratic variation A has to be understood as an integral operator on bilinear forms valued process, which can be resume as a bounded adapted $TM^{\otimes 2}$-valued process a above X and a bounded continuous increasing process ϕ :

$$A_t(h,\omega) = \int_0^t \langle h_s(\omega_s), a_s(\omega_s)\rangle\, d\phi_s, \tag{7.2.3}$$

where $h_t \in \Gamma(T^*M^{\otimes 2})$ for all t. We will use the notation dA_t for $\langle ., a_t\rangle\, d\phi_t$. The main example, linked to Brownian motions, is $dA_t(\cdot,\omega) = a\mathrm{Trace}\left(\sigma(X_t(\omega))\cdot \otimes \sigma(X_t(\omega))\cdot\right)\, dt$. Remarks that for every adapted T^*M-valued process γ, $A(\gamma\otimes .)$ is a drift. We denote by dX its Ito derivative and by $d_m^R X$ the martingale part of dX with respect to R. Both are infinitesimal vector fields. With this notation, we have :

$$dX_t = d_m^R X_t + dB_t,\ R\text{-a.s}, \tag{7.2.4}$$

and

$$d[X,X]_t = dA_t,\ R\text{-a.s.} \tag{7.2.5}$$

The Léonard's entropy condition, stronger than the usual absolutely condition in Girsanov theory, brings better boundedness properties on the drift vector field. That is why, we need to introduce the following functional spaces. Let $\mathcal{G}$ be the space of measurable functions $g : [0,1]\times\Omega \to T^*M$ such that for all $t \in [0,1]$ and $\omega \in \Omega$, $g_t(\omega) \in T^*_{\omega_t}M$. For any probability measure Q on Ω, we defined a semi-norm on $\mathcal{G}$ by

$$\|g\|_{\mathcal{G}(Q)} = \mathbb{E}_Q\left[A_1(g\otimes g)\right]^{1/2}. \tag{7.2.6}$$

Up to identification of function as class with respect to the semi-norm $\|.\|_{\mathcal{G}(Q)}$, we defined the following Hilbert spaces :

$$\mathcal{G}(Q) = \left\{ g \in \mathcal{G} : \|g\|_{\mathcal{G}(Q)} < +\infty \right\}, \tag{7.2.7}$$

and

$$\mathcal{H}(Q) = \{g \in \mathcal{G}(Q) : g \,\text{adapted}\} \tag{7.2.8}$$

Theorem 7.2.2 (Girsanov's theorem)**.** *Let P be a probability measure on Ω such that $H(P|R) < +\infty$. Then P is a law of a semi-martingale and there exists an adapted process $\beta \in \mathcal{H}(P)$ such that $P \in \mathcal{MP}(B + \hat{B}, A)$ where $\hat{B} = A(\beta \otimes .)$.*

Remark 7.2.3. *In other words, β satisfies :*

$$\mathbb{E}_P \left[\int_0^1 \langle df, d_m^R X_t - dA_t(\beta_t \otimes .) \rangle \right] = 0, \, \forall f \in \mathcal{C}_c^\infty(M). \tag{7.2.9}$$

Under the classical assumption, the Girsanov theorem prove the existence of a drift $\beta \in \mathcal{G}$ such that $A_1(\beta \otimes \beta) < +\infty$ P-a.s where our result gives finiteness in expectation.

Before beginning the proof, let us recall a well-known result about local martingales (see [71] for instance).

Lemma 7.2.4. *Let M be a positive local martingale, on some filtered probability space. Then M is a super-martingale. In addition, if M is also uniformly integrable, then it is a martingale.*

Proof. Let $(\tau_n)_n$ be a localizing sequence for M. For all $s \leq t$, with Fatou's lemma, we have :

$$\begin{aligned}
\mathbb{E}[M_t|\mathcal{F}_s] &= \mathbb{E}[\lim_{n\to\infty} M_t^{\tau_n}|\mathcal{F}_s] \\
&\leq \liminf_{n\to\infty} \mathbb{E}[M_t^{\tau_n}|\mathcal{F}_s] \\
&\leq \liminf_{n\to\infty} M_s^{\tau_n}] \\
&= M_s \,\text{a.s.}
\end{aligned}$$

So M is a super-martingale. Now, we assume that M is uniformly integrable. For all $T > 0$, we have

$$\lim_{n\to\infty} M_T^{\tau_n} = M_T \,\text{a.s.}$$

As the sequence $(M_t^{\tau_n})_n$ is uniformly integrable, by Vitali's theorem, the convergence is also in $\mathbb{L}^1$. In particular, we have $\mathbb{E}[M_T] = \mathbb{E}[M_0]$. So M is also a martingale. $\square$

Proof of Theorem 7.2.2. For $h \in \mathcal{H}(P)$, we define the processes N^h by

$$N_t^h = \int_0^t \langle h_s, d_m^R X_s \rangle,\ 0 \le t \le 1. \tag{7.2.10}$$

In restriction to $h \in \mathcal{H}(P) \cap \mathcal{H}(R)$, N^h is a stochastic integral with respect to R. Its stochastic exponential $\mathcal{E}(N^h)$ is a positive local martingale, so a super-martingale and we have :

$$0 \le \mathbb{E}_R[\mathcal{E}(N^h)_1] \le 1. \tag{7.2.11}$$

For $h \in \mathcal{H}(P) \cap \mathcal{H}(R)$, let u be the function $u : \omega \in \Omega \mapsto N_1^h - \frac{1}{2}[N^h, N^h]_1$. According to Lemma 7.2.1, $u \in \mathbb{L}^1(P)$ and we have:

$$\mathbb{E}_P[u] - \log \mathbb{E}_R[e^u] \le H(P|R) < +\infty.$$

Using equation (7.2.11), we have :

$$\mathbb{E}_P[u] \le H(P|R).$$

Then, as $\mathbb{E}_P\left[[N^h, N^h]_1\right] = \|h\|^2_{\mathcal{G}(P)}$ which is finite, we have:

$$\mathbb{E}_P[N_1^h] \le H(P|R) + \frac{1}{2}\|h\|^2_{\mathcal{G}(P)}.$$

From the same calculation, with $-h$ and λh with $\lambda > 0$, it yields :

$$\lambda \left|\mathbb{E}_P[N_1^h]\right| \le H(P|R) + \frac{\lambda^2}{2}\|h\|_{\mathcal{G}(P)},\ \forall h \in \mathcal{H}(P) \cap \mathcal{H}(R). \tag{7.2.12}$$

For $\|h\|_{\mathcal{G}} \neq 0$, we can take $\lambda = \sqrt{2H(P|R)}\|h\|_{\mathcal{G}}^{-1}$ and we obtain :

$$\left|\mathbb{E}_P[N_1^h]\right| \le \sqrt{2H(P|R)}\|h\|_{\mathcal{G}} \tag{7.2.13}$$

Letting λ tends to 0 in (7.2.12), this inequality remains valid for $\|H\|_{\mathcal{G}} = 0$. So the linear form $h \mapsto \mathbb{E}_P[N_1^h]$ is continuous on $\mathcal{H}(P) \cap \mathcal{H}(R)$. This set is dense in $\mathcal{H}(P)$ because it contains the dense set of simple functions

$$h : (t, \omega) \in [0,1] \times \Omega \mapsto \sum_{i=1}^k \sigma(X_t) h_i \mathbb{1}_{]S_i, T_i]},$$

with $k \in \mathbb{N}$, $(h_i)_{1 \le i \le k} \in \mathbb{R}^n$ and $S_i < T_i \le S_{i+1}$ stopping times.

So, there exists a unique extension as linear form on $\mathcal{H}(P)$. By Riesz representation theorem, there exists a process $\beta \in \mathcal{H}(P)$, dual of this linear form, i.e

$$\mathbb{E}_P\left[\int_0^1 \langle df, d_m^R X_t \rangle\right] = \mathbb{E}_P\left[\int_0^1 \langle df, dA_t(\beta_t \otimes .)\rangle\right]. \tag{7.2.14}$$

Then, under P, X is a semi-martingale with quadratic variation A and drift $A(\beta \otimes \cdot)$. □

The Léonard approach of Girsanov theory allow to obtain an explicit expression of the density $\frac{dP}{dR}$, under an uniqueness condition. For any probability measure $Q \in \mathcal{P}(\Omega)$, we denote by $Q_0 = X_0\#Q$ the law of the initial position X_0 under Q. For a stopping time τ, we denote X^τ the stopping process $X^\tau_t = X_{t\wedge\tau}$ and $Q^\tau = X_{[0,\tau]}\#Q$ its law.

Definition 7.2.5 (Condition (U))**.** *A measure $Q \in \mathcal{MP}(B, A)$ satisfies the uniqueness condition (U) if for any measure $Q' \in \mathcal{MP}(B, A)$ such that $Q'_0 = Q_0$ and $Q' \ll Q$ then $Q' = Q$*

The measures of solutions of a SDE with Lipschitz coefficients satisfy the condition (U). This property has some stability under stopping.

Proposition 7.2.6. *Assume that $Q \in \mathcal{MP}(B, A)$ satisfies the condition (U). For all stopping time τ, the law Q^τ fulfils (U) too.*

Proof. By assumption, $Q^\tau \in \mathcal{MP}(B, A)$. Let $P \in \mathcal{MP}(B, A)$ such that $P_0 = Q^\tau_0$ and $P \ll Q^\tau$. We want to show that $P = Q^\tau$. By disintegration, we have :

$$Q = Q_{[0,\tau]} \otimes Q\left(.|X_{[0,\tau]}\right). \tag{7.2.15}$$

We define the auxiliary measure $\tilde{P} := P_{[0,\tau]} \otimes Q\left(.|X_{[0,\tau]}\right)$. Remarks that $\tilde{P}^\tau = P$. The measure $\tilde{P}$ satisfies $\tilde{P}_0 = Q_0$ and $\tilde{P} \ll Q$. If we prove that it also satisfies the martingale problem $\mathcal{MP}(B, A)$, then the property (U) applied to Q will show that $\tilde{P} = Q$ and $P = \tilde{P}^\tau = Q^\tau$. For all $f \in \mathcal{C}^\infty_c(M)$, let $(\sigma_n)_n$ be a localizing sequence for M^f. We have :

$$\begin{aligned}
\mathbb{E}_{\tilde{P}}[f(X^{\sigma_n}_t) - f(X^{\sigma_n}_0)] = {} & \mathbb{E}_{\tilde{P}}\left[\mathbb{1}_{t\geq\tau}\left(f(X^{\sigma_n}_t) - f(X^{\sigma_n}_\tau)\right)\right] \\
& + \mathbb{E}_{\tilde{P}}\left[\mathbb{1}_{t\geq\tau}\left(f(X^{\sigma_n}_\tau) - f(X^{\sigma_n}_0)\right)\right] \\
& + \mathbb{E}_{\tilde{P}}\left[\mathbb{1}_{t<\tau}\left(f(X^{\sigma_n}_t) - f(X^{\sigma_n}_0)\right)\right].
\end{aligned}$$

The two last terms are $\mathcal{F}_\tau$-measurable and on this σ-field, $\tilde{P}$ coincides with Q. So we have :

$$\begin{aligned}
\mathbb{E}_{\tilde{P}}\left[\mathbb{1}_{t<\tau}\left(f(X^{\sigma_n}_t) - f(X^{\sigma_n}_0)\right)\right] &= \mathbb{E}_{Q}\left[\mathbb{1}_{t<\tau}\left(f(X^{\sigma_n}_t) - f(X^{\sigma_n}_0)\right)\right] \\
&= \mathbb{E}_{Q}\left[\mathbb{1}_{t<\tau}\left(f(X^{\sigma_n}_{t\wedge\tau}) - f(X^{\sigma_n}_0)\right)\right] \\
\mathbb{E}_{\tilde{P}}\left[\mathbb{1}_{t\geq\tau}\left(f(X^{\sigma_n}_\tau) - f(X^{\sigma_n}_0)\right)\right] &= \mathbb{E}_{Q}\left[\mathbb{1}_{t\geq\tau}\left(f(X^{\sigma_n}_{t\wedge\tau}) - f(X^{\sigma_n}_0)\right)\right]
\end{aligned}$$

Hence, for the sum of these two terms, we have :

$$\mathbb{E}_{\tilde{P}}\left[\mathbb{1}_{t\geq\tau}\left(f(X^{\sigma_n}_\tau) - f(X^{\sigma_n}_0)\right)\right] + \mathbb{E}_{\tilde{P}}\left[\mathbb{1}_{t<\tau}\left(f(X^{\sigma_n}_t) - f(X^{\sigma_n}_0)\right)\right]$$

$$
\begin{aligned}
&= \mathbb{E}_P\left[\left(f(X^{\sigma_n}_{t\wedge\tau}) - f(X^{\sigma_n}_0)\right)\right] \\
&= \mathbb{E}_P\left[\int_0^{t\wedge\tau} \langle df, dB_s\rangle + \frac{1}{2}\int_0^{t\wedge\tau} \langle \nabla^2 f, dA\rangle\right] \\
&= \mathbb{E}_{\tilde{P}}\left[\int_0^{t\wedge\tau} \langle df, dB_s\rangle + \frac{1}{2}\int_0^{t\wedge\tau} \langle \nabla^2 f, dA\rangle\right]
\end{aligned}
$$

For the first term, we have :

$$
\begin{aligned}
&\mathbb{E}_{\tilde{P}}\left[\mathbb{1}_{t\geq\tau}\left(f(X^{\sigma_n}_t) - f(X^{\sigma_n}_\tau)\right)\right] \\
&= \int_\Omega \mathbb{E}_Q\left[\mathbb{1}_{t\geq\tau}\left(f(X^{\sigma_n}_t) - f(X^{\sigma_n}_\tau)\right) | X_{[0,\tau]} = \eta\right] P(d\eta) \\
&= \int_\Omega \mathbb{E}_Q\left[f(X^{\sigma_n}_t) - f(X^{\sigma_n}_{t\wedge\tau})) | X_{[0,\tau]} = \eta\right] P(d\eta) \\
&= \int_\Omega \mathbb{E}_Q\left[\int_{t\wedge\tau}^t \langle df, dB_s\rangle + \frac{1}{2}\int_{t\wedge\tau}^t \langle \nabla^2 f, dA\rangle \middle| X_{[0,\tau]} = \eta\right] P(d\eta) \\
&= \mathbb{E}_{\tilde{P}}\left[\int_{t\wedge\tau}^t \langle df, dB_s\rangle + \frac{1}{2}\int_{t\wedge\tau}^t \langle \nabla^2 f, dA\rangle\right]
\end{aligned}
$$

Taking the limit of the localizing sequence as $n \to \infty$, we have :

$$
\mathbb{E}_{\tilde{P}}[f(X_t) - f(X_0)] = \mathbb{E}_{\tilde{P}}\left[\int_0^t \langle df, dB_s\rangle + \frac{1}{2}A_t(\nabla^2 f)\right]. \tag{7.2.16}
$$

This shows that $\tilde{P}$ is in $\mathcal{MP}(B, A)$ and ends the proof. □

Theorem 7.2.7. *(density $\frac{dP}{dR}$) With the notations of Theorem 7.2.2, if R satisfies the condition (U), then*

$$
\frac{dP}{dR} = \mathbb{1}_{\frac{dP}{dR}>0} \frac{dP_0}{dR_0}(X_0) \exp\left(\int_0^1 \langle \beta_t, \mathrm{d}^P_m X_t\rangle - \frac{1}{2}A_1(\beta\otimes\beta)\right). \tag{7.2.17}
$$

The proof of this theorem is divided in three parts. Firstly, we recall a well-known change of measure formula, for the stopped process. Then, we prove a weaker version of Theorem 7.2.7, in the case of equivalent measure. The proof is ended by a regularisation argument. We need to introduce some notations.

Let γ be an adapted process in T^*M such that $A_1(\gamma\otimes\gamma) < \infty$, R-a.s. We define a stochastic integral process associated to γ:

$$
N_t = \int_0^t \langle \gamma_s, d^R_m X_s\rangle,\ 0 \leq t \leq 1, \tag{7.2.18}
$$

its stochastic exponential $Z_t = \mathcal{E}(N)_t$ and the stopping times family :

$$
\sigma_k = \inf\{t \in [0,1] : A_t(\gamma\otimes\gamma) \geq k\},\ k \geq 1. \tag{7.2.19}
$$

For a probability Q, we denote by Q^k the law $X^{\sigma_k}_{\#}Q$ and Z^k the stopped process Z^{σ_k}.

Lemma 7.2.8. *For all $k \geq 1$, Z^k is a R^k martingale and the measure*

$$Q_k = Z_1^k R^k$$

is a probability measure in $\mathcal{MP}(\tilde{B}^k, A^k)$ where $\tilde{B} = B + A(\gamma \otimes .)$.

Proof. The process Z^k is a positive local martingale for R^k. From Lemma 7.2.4, Z^k is also a super-martingale. For all $p > 0$, we have :

$$\mathcal{E}(N)_t^p \leq e^{pN_t} \leq e^{kp^2/2}\mathcal{E}(pN)_t,\ R^k\text{a.s.} \tag{7.2.20}$$

As $\mathcal{E}(pN)$ is also a R^k super-martingale, we have :

$$\mathbb{E}_{R^k}\left[\mathcal{E}(N)_t^p\right] \leq e^{kp^2/2}\mathbb{E}_{R^k}\left[\mathcal{E}(pN)_t\right] \leq e^{kp^2/2} < \infty. \tag{7.2.21}$$

For $p > 1$, we can deduce that Z^k is uniformly integrable and using Lemma 7.2.4 again, Z^k is a R^k-martingale. In particular, we have : $\mathbb{E}_{R^k}[Z_1^k] = 1$. Hence, Q^k is a probability measure. The quadratic variation of X under R is A so it is A^k under R^k. As Q_k is absolutely continuous with respect to R^k, then the quadratic variation of X with respect to Q_k is A^k.

Let $f \in \mathcal{C}_c^\infty(M)$, $(\tau_n)_n$ a localizing sequence and $t \in [0,1]$, we have :

$$\begin{aligned}
\mathbb{E}_{Q_k}\left[f(X_t^{\tau_n}) - f(X_0^{\tau_n})\right] &= \mathbb{E}_{R^k}\left[Z_t^k f(X_t^{\tau_n}) - Z_0^k f(X_0^{\tau_n})\right] \\
&= \mathbb{E}_{R^k}\left[\int_0^t \left(f(X_s^{\tau_n})dZ_s^k + Z_s^k df(X_s^{\tau_n}) + d[f(X^{\tau_n}), Z^k]\right)\right] \\
&= \mathbb{E}_{R^k}\Bigg[\int_0^{t\wedge\tau_n} f(X_s)Z_s^k\langle\gamma_s, d_m^R X_s\rangle + \int_0^{t\wedge\tau_n} Z_s^k\langle df(X_s), d_m^R X_s\rangle \\
&\quad + \int_0^{t\wedge\tau_n} Z_s^k\langle df(X_s), dB_s^k\rangle + \frac{1}{2}\int_0^{t\wedge\tau_n} Z_s^k dA_s^k(\nabla^2 f) \\
&\quad + \int_0^{t\wedge\tau_n} Z_s^k dA_s^k(\gamma \otimes df(X))\Bigg] \\
&= \mathbb{E}_{R^k}\Bigg[\int_0^{t\wedge\tau_n} Z_s^k\langle df(X_s), dB_s^k\rangle + \frac{1}{2}\int_0^{t\wedge\tau_n} Z_s^k dA_s^k(\nabla^2 f) \\
&\quad + \int_0^{t\wedge\tau_n} Z_s^k dA_s^k(\gamma \otimes df(X))\Bigg] \\
&= \mathbb{E}_{R^k}\Bigg[Z_{t\wedge\tau_n}^k \int_0^t \langle df(X_s^{\tau_n}), dB_s^k\rangle + \frac{1}{2}Z_{t\wedge\tau_n}^k A_{t\wedge\tau_n}^k(\nabla^2 f) \\
&\quad + Z_{t\wedge\tau_n}^k A_{t\wedge\tau_n}^k(\gamma \otimes df(X))\Bigg]
\end{aligned}$$

Taking the limit in the localising sequence, we have :

$$\mathbb{E}_{Q_k}\left[f(X_t) - f(X_0)\right] = \mathbb{E}_{Q_k}\left[\int_0^t \langle df(X_s), dB_s^k + A_s^k(\gamma \otimes .)\rangle + \frac{1}{2}A_t^k(\nabla^2 f)\right].$$

So we proved that $Q_k \in \mathcal{MP}(\tilde{B}^k, A^k)$. $\square$

Remark 7.2.9. *As seen in the proof, this lemma remains true for any measure R in any $\mathcal{MP}(B', A')$ and not only our reference measure R.*

Now, we have the tools to prove a weak version of Theorem 7.2.7, under absolutely continuous condition.

Lemma 7.2.10. *Let P and β be as in Theorem 7.2.7 and assume that $P \sim R$. Then for all $k \geq 1$, we have :*

$$\frac{dP}{dR}(X) = \frac{dP_0}{dR_0}(X_0)\exp\left(\int_0^1 \langle \beta_t, \mathrm{d}_m^R X_t\rangle - \frac{1}{2}A_1(\beta \otimes \beta)\right), \ R\text{-}a.s.$$

Proof. By conditioning with respect to the initial position X_0, we can assume that $P_0 = R_0$. For all $k \geq 1$, we define the stopping time

$$\tau_k = \inf\left\{t \in [0,1] : A_t(\beta \otimes \beta) > k\right\}$$

and the measures

$$\begin{aligned}
Q_k &= \mathcal{E}\left(\int_0^\cdot \langle -\beta_s, d_m^R X_s\rangle\right)_{\tau_k \wedge 1} P^k, \\
\tilde{P}^k &= \mathcal{E}\left(\int_0^\cdot \langle \beta_s, d_m^R X_s\rangle\right)_{\tau_k \wedge 1} R^k, \\
\tilde{Q}_k &= \mathcal{E}\left(\int_0^\cdot \langle -\beta_s, d_m^R X_s\rangle\right)_{\tau_k \wedge 1} \tilde{P}^k.
\end{aligned}$$

The idea of the roof is to give two different expressions of R^k, using the uniqueness condition (U) and to compare these expressions. Remark that $\beta \in \mathcal{H}(P)^*$ is stronger than $A_1(\beta \otimes \beta) < \infty$, so we can use Lemma 7.2.8 and these measure are probability measures. More precisely, Q_k is a probability in $\mathcal{MP}(B^k, A^k)$ and using the condition (U) for R and Lemma 7.2.6, we have : for all $k \geq 1$, $R^k = Q_k$. Similarly, we have $\tilde{P}^k \in \mathcal{MP}(B^k + \hat{B}^k, A^k)$ and $\tilde{Q}_k \in \mathcal{MP}(B^k, A^k)$. Using again the condition (U), we have : $\tilde{Q}_k = R^k$. So, we have $Q^k = \tilde{Q}^k$. Since the stochastic exponential is positive, this means that : $P^k = \tilde{P}^k$, i.e

$$P^k = \mathcal{E}\left(\int_0^\cdot \langle \beta_s, \mathrm{d}_m^R X_s\rangle\right)_{\tau_k \wedge 1} R^k, \ \forall k \geq 1. \tag{7.2.22}$$

So we have :

$$\mathbb{1}_{[0,\tau_k\wedge 1]}\frac{dP^k}{dR^k}(X) = \mathbb{1}_{[0,\tau_k\wedge 1]}\exp\left(\int_0^1 \langle \beta_t, \mathrm{d}_m^R X_t\rangle - \frac{1}{2}A_1(\beta\otimes\beta)\right), \ R\text{-a.s.} \quad (7.2.23)$$

We denote by τ the limit of τ_n. As $\beta \in \mathcal{H}(P)$, $\tau = \infty$ P-a.s and as P and R are equivalent, $\tau = \infty$ R-a.s too. Then, for R almost ω, there exists a k_0 such that $\tau_{k_0}(\omega) > 1$. Using the previous equation with $k = k_0$, we have :

$$\frac{dP}{dR}(\omega) = \exp\left(\int_0^1 \langle \beta_t, \mathrm{d}_m^R X_t\rangle - \frac{1}{2}A_1(\beta\otimes\beta)\right)(\omega). \quad (7.2.24)$$

□

Remark that Until equation (7.2.23), the assumption $P \sim R$ is not used. We need it to defined the stopping times τ_k R-a.s. To conclude the proof of Theorem 7.2.7, we use a regularization argument : we prove the result for a sequence of measures absolutely continuous with respect to R, which converges to P.

Proof of Theorem 7.2.7. For $n \geq 1$, we consider the probability

$$P_n = \left(1 - \frac{1}{n}\right)P + \frac{1}{n}R. \quad (7.2.25)$$

For all $n \geq 1$, we have $P_n \sim R$ and using the convexity of the entropy, we have : $H(P_n|R) \leq H(P|R) < \infty$. Let us prove that P_n converges to P. It can be checked that $\mathbb{1}_{\frac{dP}{dR}\geq 1}\log\left(\frac{dP}{dP_n}\right)$ and $\mathbb{1}_{\frac{dP}{dR}< 1}\log\left(\frac{dP}{dP_n}\right)$ are respectively decreasing and increasing sequence of functions. By the monotone convergence theorem, we have :

$$\begin{aligned}\lim_{n\to\infty} H(P|P_n) &= \lim_{n\to\infty}\int_\Omega \log(\frac{dP}{dP_n})\, dP \\ &= \lim_{n\to\infty}\int_{\frac{dP}{dR}\geq 1}\log(\frac{dP}{dP_n})\, dP + \lim_{n\to\infty}\int_{\frac{dP}{dR}< 1}\log(\frac{dP}{dP_n})\, dP \\ &= 0\end{aligned}$$

By Theorem 7.2.2, there exists vector fields β and β^n defined P-a.s and R-a.s respectively, such that $\mathbb{E}_P[A_1(\beta\otimes\beta)] < \infty$, $\mathbb{E}_{P_n}[A_1(\beta^n\otimes\beta^n)]$, $P \in \mathcal{MP}(B+\hat{B}, A)$ and $P_n \in \mathcal{MP}(B + \hat{B}^n, A)$, where $\hat{B}^n_t = A_t(\beta^n \otimes .)$. In Itô's notation, we have :

$$d_m^{P_n}X_t = d_m^P X_t + dA_t\left((\beta - \beta^n)\otimes .\right). \quad (7.2.26)$$

We extend arbitrarily β by 0 on the P-null set to have a process defined R-a.s. It is then possible to consider the P_n local martingale

$$Y_t = \mathcal{E}\left(\int_0^{\cdot}\langle \beta_s - \beta^n_s, d_m^{P_n}X_s\rangle\right)_t. \quad (7.2.27)$$

From Lemma 7.2.4, Y is a P_n super-martingale so $\mathbb{E}_{P_n}[Y_1] \leq 1$. Using a last time the variational view of the entropy 7.2.1, we have :

$$\begin{aligned}
H(P|P_n) &\geq \mathbb{E}_P\left[\int_0^1 \left\langle (\beta_t - \beta_t^n, d_m^{P_n} X_s\right\rangle - \frac{1}{2} A_1\left((\beta - \beta^n) \otimes (\beta - \beta^n)\right)\right] - \log \mathbb{E}_{P_n}[Y_1] \\
&\geq \mathbb{E}_P\left[\int_0^1 \left\langle (\beta_t - \beta_t^n, d_m^{P} X_s + dA_s\left((\beta - \beta^n) \otimes .\right)\right\rangle\right. \\
&\qquad \left. - \frac{1}{2} A_1\left((\beta - \beta^n) \otimes (\beta - \beta^n)\right)\right] \\
&\geq \frac{1}{2} \mathbb{E}_P\left[A_1\left((\beta - \beta^n) \otimes (\beta - \beta^n)\right)\right].
\end{aligned}$$

Together with the entropy convergence, we obtain the estimation :

$$\lim_{n \to \infty} \mathbb{E}_P[A_1\left((\beta - \beta^n) \otimes (\beta - \beta^n)\right)] = 0. \tag{7.2.28}$$

As $P_n \to P$ in total variation convergence, up to extraction, we have $\frac{dP_n}{dR} \to \frac{dP}{dR}$ and $\frac{dP_{n,0}}{dR} \to \frac{dP_0}{dR}$ R-a.s. Besides, the estimation (7.2.28) prove the convergence P almost surely

$$\exp\left(\int_0^1 \langle \beta_t^n, \mathrm{d}_m^R X_t\rangle - \frac{1}{2} A_1(\beta^n \otimes \beta^n)\right) \to \exp\left(\int_0^1 \langle \beta_t, \mathrm{d}_m^R X_t\rangle - \frac{1}{2} A_1(\beta \otimes \beta)\right). \tag{7.2.29}$$

This ends the proof. □

As corollary, we have a link between the entropy $H(P|R)$ and the norm $\|\beta\|_{\mathcal{G}(P)}$.

Corollary 7.2.11. *With the assumptions of Theorem 7.2.4, we have :*

$$H(P|R) = H(P_0|R_0) + \frac{1}{2} \|\beta\|_{\mathcal{G}(P)}.$$

Proof. From equation (7.2.23), we have :

$$\begin{aligned}
H(P^k|R^k) &= E_{P^k}[\log \frac{dP^k}{dR^k}] \\
&= E_P[\log \frac{dP_0}{dR_0}] + \mathbb{E}_{P^k}[\int_0^1 \langle \beta_s, d_m^R X_s\rangle - \frac{1}{2} A_1(\beta \otimes \beta)] \\
&= H(P_0|R_0) + \mathbb{E}_{P^k}[\int_0^1 \langle \beta_s, d_m^R X_s - dA_s(\beta \otimes \cdot)\rangle + \frac{1}{2} A_1(\beta \otimes \beta)] \\
&= H(P_0|R_0) + \frac{1}{2} \mathbb{E}_P[A_{1 \wedge \tau_k}(\beta \otimes \beta)]
\end{aligned}$$

By monotonicity, the expectation on the right side converges to $\|\beta\|_{\mathcal{G}(P)}$. Using a variational characterisation of the entropy, the left side converges to $H(P|R)$. □

7.3 Semi-martingale properties

In this section, we apply the Girsanov theory of Section 7.2 to Brenier-Schrödinger problem. We obtain a characterisation of solution as semi-martingales and an expression of the density with respect to R. The notion of stochastic velocity, introduced by Nelson in [65] make a link between minimization of entropy and minimization of some kinetic energy, as in the Brenier problem.

The reference measure of Brenier-Schrödinger problem is the reversible reflected Brownian motion defined in (6.5.2). Using notations of the previous section, it is a semi-martingale measure with drift

$$dB_t(\omega) = \nu_{\omega_t} dL_t(\omega)$$

and quadratic variation

$$dA_t(\cdot, \omega) = a\mathrm{Trace}\left(\sigma(\omega_t) \cdot \otimes \sigma(\omega_t) \cdot\right)\, dt$$

for all $\omega \in \Omega$ and $t \in [0,1]$. As an application to Theorem 7.2.2, we have the first characterisation as semi-martingale of solutions.

Corollary 7.3.1. *Let P be the solution of (BS). Then P is a law of a semi-martingale and there exists an adapted process $\beta \in \mathcal{H}(P)$ such that*

$$P \in \mathcal{MP}(\nu dL_t + a\beta_t dt, A).$$

Actually, according to Theorem 7.2.2, β is a T^*M-valued process whereas the drift $A(\beta \otimes \cdot)$ is a TM-valued process. The formula

$$dA_t(\beta \otimes \cdot, \omega) = \sum_i a\langle \beta_t(\omega), \sigma_{\omega_t}(e_i)\rangle \sigma_{\omega_t}(e_i) dt,\ \forall t \in [0,1],\ \forall \omega \in \Omega, \tag{7.3.1}$$

Gives an identification between T^*M and TM and from now, β will be interpreted as a drift operator. It is the velocity part of the drift. Nelson introduced in [65] the notion of stochastic velocity (also known as mean derivative) for real process. A generalisation of this notion to manifold can be found in [43]. Let $t \in [0,1]$, X_t does not belong to ∂M P-a.s. So the exponential map is a local diffeomorphism between a neighbourhood $N(X_t) \subset T_{X_t}M$ and $U(X_t) \subset M$ P-a.s and the stopping time

$$\tau_t = \inf\left\{h \geq 0 : X_{t+h} \in U(X_t)\right\} \tag{7.3.2}$$

is positive. It allows us to define mean derivatives in a manifold with boundary. For $x \in M$ and $y \in U(x)$, $\overrightarrow{xy}$ denotes the vector $\log_x(y) \in T_xM$.

Definition 7.3.2. *The forward stochastic velocity of P is the adapted vector field process $\overset{\rightharpoonup P}{v} : [0,1] \times \Omega \to TM$ defined by :*

$$\overset{\rightharpoonup P}{v}_t = \lim_{h \to 0^+} \frac{1}{h} \mathbb{E}_P \left[\overrightarrow{X_t X_{t+h\wedge\tau_t}} | X_{[0,t]} \right],$$

providing that the following limits exists in some sense.

In the case of [5], the drift of P is equal to is forward stochastic velocity. In our case, the reflection part plays apart.

Proposition 7.3.3. *The measure P admit a forward stochastic velocity and we have :*

$$\overset{\rightharpoonup P}{v}_t = a\beta_t,\ P \otimes dt\text{-a.s.} \tag{7.3.3}$$

Proof. From Corollary 7.3.1, we have : $dX_t = d_m^P X_t + a\beta_t dt + \nu_{X_t} dL_t$ P-a.s. Let $t \geq 0$ and $\eta \in \mathcal{C}([0,t], M)$ such that $x = \eta_t \notin \partial M$. Applying Ito formula to $\log_x$, we have :

$$\begin{aligned} \mathbb{E}_P \left[\overrightarrow{X_t X_{t+h\wedge\tau_t}} | X_{[0,t]} = \eta \right] =& \mathbb{E}_P \left[\int_0^{h\wedge\tau} \langle d\log_x(X_s), a\beta_s(X) \rangle \, ds \right. \\ & \left. + \int_0^{h\wedge\tau} \frac{a}{2} \Delta \log_x(X_s) \, ds \right| X_{[0,t]} = \eta \Bigg]. \end{aligned}$$

The inversion of integral and limits are guaranteed by the compactness of M and regularity of $\log_x$. Besides, from Proposition, 2.1.4, we have $d\log_x(x) = \mathrm{id}_{T_x M}$ and $\Delta \log_x(x) = 0$. This ends the proof. □

This result could also be applied to the measures R^x and P^x. It proves that the velocities $\overset{x\rightharpoonup}{v}$, defined as the stochastic velocities of P^x, are equals to $a\beta^x$ where $\beta^x \in \mathcal{H}(P^x)$. Using the measurability of Doob-Meyer decomposition (see [67] for instance) and the disintegration of P with respect to the initial position, it turns out that

$$\overset{\rightharpoonup P}{v} = \overset{X_0 \rightharpoonup}{v} \quad P\text{-a.s.} \tag{7.3.4}$$

We now are interested to the density of the solution P with respect to R. For that, we need to check that R satisfies the condition (U). Actually, from the uniqueness of solution in the Skorokhod problem (6.5.2), R verifies this condition and we have the following corollary.

Corollary 7.3.4. *The density of P is given by*

$$\frac{dP}{dR} = \mathbb{1}_{\{\frac{dP}{dR} > 0\}} \frac{dP_0}{dR_0}(X_0) \exp\left(\int_0^1 \langle \beta_t, d_m^P X_t \rangle - \frac{a}{2} \int_0^1 |\beta_s|^2 \, ds \right).$$

This formula establish a link between the velocity and the density. Remark that the reflection part does not play a role in the density. It will be very useful in the kinematic study of the following chapter. We finish this section with the interpretation of Corollary 7.2.11 for the Brenier-Schrödinger problem.

Corollary 7.3.5. *The relative entropy of the solution P is given by :*

$$H(P|R) = H(\pi_0|R_0) + \frac{1}{2a}\mathbb{E}_P\left[\int_0^1 |\overrightarrow{v}_t^P|^2\, dt\right].$$

This link between entropy and stochastic velocity is known for a long time (see [77]). The interpretation of this formula as a parallel between the Brenier problem and the Brenier-Schrödinger problem can be found in [5]. Indeed, the Brenier problem is a minimisation of a classical kinetic energy : $\mathbb{E}[\int_0^1 \dot{X}_t^2\, dt]$. Here, the minimisation of entropy is equivalent to minimization of a stochastic kinetic energy. The entropy formulation of the problem allows us to use convex optimisation tools but to understand the problem, we must keep in mind that it is a Lagrangian minimisation.

7.4 Reciprocal solution

After the semi-martingale point of view, in this last section, we present the characterisation of a solution as reciprocal measure. The goal of this section is to complete the characterisation of the solution and to prepare the study of its velocity in Chapter 8. Results presented in this section come from [5] and [16]. Before the main theorem, let us recall some vocabulary.

Definition 7.4.1. *A path measure $Q \in \mathcal{P}(\Omega)$ is called reciprocal if*

- *Q_t is σ-finite for all $t \in [0,1]$,*
- *for any $0 \leq s \leq u \leq 1$ and $A \in \sigma(X_{(0,s)})$, $B \in \sigma(X_{(s,u)})$ and $C \in \sigma(X_{(u,1)})$,*

$$Q(A \cap B \cap C|X_s, X_u) = Q(A \cap C|X_s, X_u)Q(B|X_s, X_u).$$

Intuitively, this property says that conditionally to the time s an t, events in $[s,t]$ and events outside $]s,t[$ are independent. It is a weaker property than being a Markov measure. If Q is reciprocal, then $Q(\cdot|X_0)$ and $Q(\cdot|X_1)$ are Markov measures. Indeed, if we denote $\tilde{Q} = Q(\cdot|X_0)$, for $t \geq 0$, $A \in \sigma(X_{[0,t]})$ and $b \in \sigma_{[t,1]})$, we have :

$$\tilde{Q}(A \cap B|X_t) = Q(A \cap B|X_0, X_t)$$

$$\begin{aligned}&\overset{(a)}{=} Q(A|X_0, X_t)Q(B|X_0, X_t)\\&= \tilde{Q}(A|X_t)\tilde{Q}(B|X_t)\end{aligned}$$

where we use the definition of reciprocal measure in (a). Then, $\tilde{Q}$ is Markovian. Reciprocal measures are particular cases of conditionable path measures : a good class of measure for which we can defined conditional expectation (see [56]).

The second notion we need to introduce is the additive functional.

Definition 7.4.2. *A measurable function $\mathcal{A}_{[0,1]} : \Omega \to [-\infty[$ is an additive functional if for any finite partition $[0,1] = \bigsqcup_k I_k$ with intervals, there exists functions $\mathcal{A}_k$, $\sigma(X_{I_k})$-measurable such that*

$$\mathcal{A}_{[0,1]} = \sum_k \mathcal{A}_k(X_{I_k})\; R\text{-}a.s.$$

in the following we will denote $\mathcal{A}_{[0,1]} = \mathcal{A}(X_{[0,1]})$. An example of such functional is given by an integral :

$$\mathcal{A}(X_{[0,1]}) = \int_{[0,1]} p_t(X_t)\, dt, \tag{7.4.1}$$

for some $p : [0,1] \times \Omega \to [-\infty, +\infty[$.

Theorem 7.4.3 ([5])**.** *The solution P is reciprocal and there exist an $[-\infty, +\infty[$-valued $\sigma(X_{\mathcal{T}})$-measurable additive functional $\mathcal{A}(X_{\mathcal{T}})$ and a measurable function η such that*

$$P = \exp(\mathcal{A}(X_{\mathcal{T}}) + \eta(X_0, X_1))R,$$

with the convention $\exp(-\infty) = 0$.

In the particular case where $\mathcal{T}$ is a finite set, it is also shown in [16] that $\mathcal{A}$ has the integral form :

$$\mathcal{A}(X_{\mathcal{T}}) = \sum_{s \in \mathcal{T}} \theta_s(X_s), \tag{7.4.2}$$

where for all $s \in \mathcal{T}$, $\theta_s : m \to [-\infty, +\infty[$ is measurable. This discrete case can be interesting when it come to simulations. This problem have been broached by [17] in toruses, with Sinkhorn algorithm. Their proof implies some Gaussian estimates of the heat kernel, satisfied in a space as $\mathbb{T}^n$.

Chapter 8

Kinematic of regular solutions

We derived the equation satisfied by the stochastic velocities of a solution. We partially recover Navier-Stokes equations for the backward velocity and the incompressibility for the current velocity.

Contents

8.1 Introduction

In this chapter, we assume that the problem (BS) admit a regular solution and we study the kinematic characteristic of such a measure. As seen in the previous chapter 7, a solution of (BS) can be determined either by its drift as a semi-martingale (it is the Girsanov point of view), or by a additive functional $\mathcal{A}(X_{\mathcal{T}})$ and some function η as the reciprocal measure $\exp(\mathcal{A}(X_{\mathcal{T}}) + \eta(X_0, X_1))$ (it is the reciprocal measure point of view). We want to make a link between these visions under some regularity assumption. In the case of the Brenier problem, the solution satisfies,in some sense, the Euler equation. The goal of this chapter is to show that some characteristic quantities satisfy the Navier-Stokes equation. It extends the work in [5] which study the kinematic in $\mathbb{R}^n$ and in torus $\mathbb{T}^n$. The main originality is the behaviour of velocity at the boundary.

Let us summarise this chapter. In Section 8.2, we explain the regularity assumptions we add to our problem and we introduce the notion of regular solution. In Section 8.3, we derive the equation satisfied by the forward stochastic velocity. This characteristic quantity turns out to be not very relevant to fulfil our goal. In Section 8.4, we defined the notion of backward stochastic velocity. The link between forward and backward velocity allows us to prove that it partially satisfies Navier-Stokes equation. We finish in Section 8.5 with continuity equation. We have to introduce a new characteristic quantity : the current velocity.

8.2 Regular solutions

In this section, we introduce the notion of regular solution. Firstly, we need some regularity on the BS problem itself. Let $\mathcal{T} \subset [0,1]$ be a finite union of intervals and $\mathcal{S}$ finite subset of $]0,1[$. This make a partition between regular times and shocking times. We are looking to the refined Brenier-Schrödinger problem :

$$H(Q|R) \to \min,\ Q \in \mathcal{P}(\Omega),\ [Q_t = \mu_t,\ \forall t \in \mathcal{T} \cup \mathcal{S}],\ Q_{01} = \pi, \tag{8.2.1}$$

with the usual assumption on $(\mu_t)_{t \in \mathcal{T} \cup \mathcal{S}}$. A regular solution of the (BS) problem is a solution which can be written as

$$P = \exp\left(\eta(X_0, X_1) + \sum_{s \in \mathcal{S}} \theta_s(X_s) + \int_{\mathcal{T}} p_t(X_t)\, dt\right) R. \tag{8.2.2}$$

for some functions $\eta : M^2 \to \mathbb{R}$, $p : \mathcal{T} \times M \to \mathbb{R}$ and $\theta_s : M \to \mathbb{R}$ for all $s \in \mathcal{S}$ such that the functions ψ^x defined in 8.3.1 below is well-defined, $\mathcal{C}^2$ in space and piecewise $\mathcal{C}^1$ in time. The specific form of $\mathcal{A}$ comes from a dual-primal problem result explained in [5]. The function p as to be seen as a pressure field and the functions θ_s as shock potentials. There is no general result for the existence of pressure field but we can cite the recent work of [15] on the existence of a pressure for the Brenier-Schrödinger problem in the n-dimensional torus with the constraint $\mu_t = \mathrm{vol}$ for all $t \in [0,1]$. An other close result of existence has been cited in Chapter 7 for the discrete problem. The existence of a regular solution P is an additional assumption we make for the following sections.

8.3 Forward kinematic

In this section, we calculate the equation satisfied by the forward stochastic $\overrightarrow{v}^P$. We can foresee from [5] that this velocity will not fulfil all our expectation but this calculation prepare the study of the backward velocity. From Corollary 7.3.1, we recall that the forward stochastic velocity has the form $\overrightarrow{v}^P = a\beta^P \in \mathcal{H}(P)$.

Lemma 8.3.1. *The vector field β^P satisfies*

$$\langle \beta_t^P, d_m^R X_t\rangle - \frac{a}{2}\left|\beta_t^P\right|^2 dt = \mathbb{1}_{\mathcal{S}}(t)\theta_t + p_t dt + d\psi_t^{X_0}(X_t),\ 0 \le t \le 1,\ P\text{-a.s}$$

where the function ψ is defined for any $t \in [0,1]$ and $x, z \in M$ by

$$\psi_t^x(z) = \log \mathbb{E}_{R^x}\left[exp\left(\eta(x, X_1) + \sum_{s\in\mathcal{S}, s>t} \theta_s(X_s) + \int_{\mathcal{T}\cap]t,1]} p_r(X_r)\, dr\right)\middle| X_t = z\right]. \tag{8.3.1}$$

Proof. The idea of the proof is to compare two expressions of the density of P with respect to R : the first given by Girsanov's theorem and the second by the definition of regular solutions. There is two steps for comparing these expressions. First, we disintegrate the measures with respect to the initial position so as to work with Markov measures. Then we calculate the restriction of the density to the σ-field $\mathcal{F}_t = \sigma(X_u / 0 \le u \le t)$:

$$\frac{dP^x_{[0,t]}}{dR^x_{[0,t]}} = \mathbb{E}_{R^x}\left[\frac{dP^x}{dR^x}\middle| \mathcal{F}_t\right]. \tag{8.3.2}$$

On one hand, from Corollary 7.3.1, the density is :

$$\frac{dP}{dR} = \frac{dP_0}{dR_0}(X_0)\exp\left(\int_{[0,1]} \langle \beta^P, d_m X_t\rangle - \frac{a}{2}\int_{[0,1]} \left|\beta_t^P\right|^2 dt\right),\ P\text{-a.s.} \tag{8.3.3}$$

Then, by restriction to $\mathcal{F}_t$, for all $0 \le t \le 1$ and P_0-almost $x \in M$, we have :

$$\frac{dP^x_{[0,t]}}{dR^x_{[0,t]}} = \exp\left(\int_0^t \langle \beta_s^x, d_m X_s\rangle - \frac{a}{2}\int_0^t |\beta_s^x|^2\, ds\right),\ P^x\text{-a.s.} \tag{8.3.4}$$

On the other hand, from the definition of regular solutions, we know that P has the form (8.2.2). By disintegration, for R_0-almost $x \in M$, we have :

$$\frac{dP^x}{dR^x} = \exp\left(\eta(x, X_1) + \sum_{s\in\mathcal{S}} \theta_s(X_s) + \int_{\mathcal{T}} p_t(X_t)\, dt\right),\ R^x\text{-a.s.} \tag{8.3.5}$$

Then, conditioning with respect to $\mathcal{F}_t$ and using the Markov property of R^x, we have :

$$\begin{aligned}\frac{dP^x_{[0,t]}}{dR^x_{[0,t]}} &= \mathbb{E}_{R^x}\left[\exp\left(\eta(x, X_1) + \sum_{s\in\mathcal{S}} \theta_s(X_s) + \int_{\mathcal{T}} p_r(X_r)\, dr\right)\middle| \mathcal{F}_t\right] \\ &= \exp\left(\sum_{s\le t} \theta_s(X_s) + \int_{\mathcal{T}\cap[0,t]} p_r(X_r)\, dr\right)\end{aligned}$$

$$\times\, \mathbb{E}_{R^x}\left[\exp\left(\eta(x,X_1)+\sum_{s>t}\theta_s(X_s)+\int_{\mathcal{T}\cap]t,1]}p_r(X_r)\,dr\right)\middle|\,\mathcal{F}_t\right]$$

$$= \exp\left(\sum_{s\in\mathcal{S},s\le t}\theta_s(X_s)+\int_{\mathcal{T}\cap[0,t]}p_r(X_r)\,dr+\psi_t^x(X_t)\right),\ R^x\text{-a.s.}$$

Now, we can confront both expressions of the differential of $\frac{dP^x_{[0,t]}}{dR^x_{[0,t]}}$ and we obtain, for P_0-almost $x \in M$:

$$\beta_t^x d_m^{R^x} X_t - \frac{a}{2}\,|\beta_t^x|^2\,dt = \mathbb{1}_{\mathcal{S}}(t)\theta_t + p_t dt + d\psi_t^x(X_t),\ 0\le t\le 1,\ P^x\text{-a.s.} \tag{8.3.6}$$

as required. □

Theorem 8.3.2. *Assume that P is regular with parameters η, θ and p, then for P_0-almost $x \in M$, the function ψ^x defined by 8.3.1 is a classical solution of the second-order Hamilton-Jacobi equation :*

$$\begin{cases} [\partial_t\psi^x - \frac{a}{2}\Delta\psi^x + a|\nabla\psi^x|^2 + \mathbb{1}_{\mathcal{T}}(t)p](t,z) = 0, & 0\le t<1, t\notin\mathcal{S}, z\in M,\\ \psi^x(t,.)-\psi^x(t^-,.) = -\theta_t(.), & t\in S,\\ \langle\nabla\psi^x(.,z),\nu(z)\rangle = 0, & z\in\partial M,\\ \psi^x(1,.) = \eta(x,.), & t=1.\end{cases} \tag{8.3.7}$$

Besides, the velocity vector field satisfies :

$$\overrightarrow{v}_t^P(X) = a\nabla\psi_t^{X_0}(X_t),\ P\text{-}a.s. \tag{8.3.8}$$

where $\nabla\psi_t^{X_0}(X_t)$ stands for $\nabla_z\psi_t^x(z)|_{x=X_0,z=X_t}$.

Proof. According to Corollary 7.3.1, we have : $d_m^P X_t = d_m^R X_t - a\beta_t dt$ P-a.s. So, disintegrating with respect to X_0, Lemma 8.3.1 gives for P_0-almost $x\in M$ and all $0\le t\le 1$:

$$d\psi_t^x(X_t) = \langle\beta_t^x, d_m^{P^x}X_t\rangle + \left(\frac{a}{2}|\beta_t^x|^2 - p_t(X_t)\right)dt - \mathbb{1}_{\mathcal{S}}(t)\theta_t(X_t),\ P^x\text{-a.s.} \tag{8.3.9}$$

On the other hand, with our regularity assumptions, the semi-martingale $(\psi^x(X_t)_t$ satisfies the Itô formula. For all $0\le t\le 1$, xe have :

$$\begin{aligned} d\psi_t^x(X_t) =& [\psi_t^x - \psi_{t^-}^x](X_t) + \left\langle\nabla\psi_t^x(X_t), d_m^{P^x}X_t\right\rangle + \langle\nabla\psi_t^x(X_t), a\beta_t^x\rangle\,dt \\ &+ \langle\nabla\psi_t^x(X_t),\nu_{X_t}\rangle\,dL_t + \left(\frac{a}{2}\Delta+\partial_t\right)\psi_t^x(X_t)dt,\ P^x\text{-a.s.}\end{aligned}$$

The Doob-Meyer decomposition of semi-martingale allows the identifications in previous equations :

$$\begin{cases} \beta_t^x = \nabla\psi_t^x(X_t), & dtP^x\text{-a.s.} \\ -\mathbb{1}_{\mathcal{S}}(t)\theta_t(X_t) = [\psi_t^x - \psi_{t^-}^x](X_t), & P^x\text{-a.s.} \\ \frac{a}{2}|\beta_t^x|^2 - p_t(X_t) = \langle\nabla\psi_t^x(X_t), a\beta_t^x\rangle + \left(\frac{a}{2}\Delta + \partial_t\right)\psi_t^x(X_t), & dtP^x\text{-a.s.} \\ \langle\psi_t^x(X_t), \nu_{X_t}\rangle\mathbb{1}_{\partial M}(X_t) = 0, & dtP^x\text{-a.s.} \end{cases} \tag{8.3.10}$$

As the function ψ^x is assumed regular, we obtain :

$$\begin{cases} \beta_t^P = \nabla\psi^{\omega_0}(\omega_t), & 0 \le t \le 1, \omega \in \Omega \\ -\theta_t(z) = [\psi_t^x - \psi_{t^-}^x](z), & t \in \mathcal{S}, x, z \in M \\ \left(\frac{a}{2}\Delta + \partial_t\right)\psi_t^x(z) + \frac{a}{2}|\nabla\psi_t^x(z)|^2 + \mathbb{1}_{\mathcal{T}}(t)p_t(z) = 0, & t \in [0,1[\backslash\mathcal{S}, x, z \in M \\ \langle\psi_t^x(z), \nu_z\rangle = 0, & 0 \le t \le 1, x \in M, z \in \partial M \end{cases} \tag{8.3.11}$$

This ends the proof. □

The function ψ^x plays the role of a scalar potential of the stochastic velocity $\overset{x\rightharpoonup}{v}$. By definition, we know that $\overset{\rightharpoonup}{v}^P \in \mathcal{H}(P)$ is adapted but we now have a much stronger result :

$$\overset{\rightharpoonup}{v}_t^P(X) = \overset{\rightharpoonup}{v}_t^P(X_0, X_t). \tag{8.3.12}$$

The equation satisfied by $\overset{\rightharpoonup}{v}^P$ is obtained by taking the gradient in (8.3.7). We denote by δ the adjoin of the exterior differential in L^2. The Hodge-de Rham Laplacian, denoted by $\Box$ is defined as $-(d\delta + \delta d)$. It satisfies the classical commutation formula :

$$d\Delta = \Box d.$$

This relation explains why $\Box$ arises as a natural Laplacian on vector fields.

Corollary 8.3.3. *For P_0 almost all $x \in M$, the forward stochastic velocity $\overset{x\rightharpoonup}{v}$ satisfies :*

$$\begin{cases} \left(\partial_t + \nabla_{\overset{x\rightharpoonup}{v}}\right)\overset{x\rightharpoonup}{v} = -\frac{a}{2}\Box\overset{x\rightharpoonup}{v} - \mathbb{1}_{\mathcal{T}}\nabla ap, & 0 \le t < 1, t \notin \mathcal{S}, z \in M, \\ \overset{x\rightharpoonup}{v}_t - \overset{x\rightharpoonup}{v}_{t^-} = -\theta_t(.), & t \in S, Z \in M \\ \langle\overset{x\rightharpoonup}{v}, \nu(z)\rangle = 0, & z \in \partial M, \\ \overset{x\rightharpoonup}{v}_1 = \nabla\eta(x,.), & z \in M. \end{cases} \tag{8.3.13}$$

This result need some remarks. The first equation has to be interpreted as Newton second law applied to a system. The second equation describes the evolution at shock times. The third describe the behaviour at the boundary of our domain. the fourth is the initial condition of the problem. Tho begin with the good point, the third equation in (8.4.3) shows that the velocity is tangent to the boundary of our domain. So the forward stochastic velocity $\overrightarrow{v}$ satisfies the impermeability condition. Now, looking at the first equation of the system, we recognized a convective acceleration term and a gradient of a pressure field but the term $a\Box$ appears with the wrong sign. Then it cannot be interpreted as the action of a viscous force. Furthermore, $\overset{x}{\overrightarrow{v}}$ does not seem to satisfy any continuity equation. Then it appears that the forward velocity is not the most relevant quantity to study. Remark that for any measure Q defined as (8.2.2), the previous results are still satisfied, with the corresponding parameters η, θ and p. This makes the previous calculation useful for the study another characteristic velocity.

8.4 Backward kinematic

In this section, we introduce the backward velocity. It has been introduced as the forward one by Nelson. It is an mean velocity, knowing the future. The idea is to proceed to a time inversion. As a by-product of the previous section, we show that this velocity satisfies the Newton part of Navier-Stokes equation.

Definition 8.4.1. *Using the notations of Definition 7.3.2, the backward stochastic velocity* $\overleftarrow{v}^P$ *is the process defined by :*

$$\overleftarrow{v}^P_t = \lim_{h\to 0^+} \frac{1}{h}\mathbb{E}_P\left[\overrightarrow{X_{t-h\wedge\tau_t}X_t}|X_{[t,1]}\right],$$

providing that the following limits exists in some sense.

Similarly as for the forward velocity, we have a disintegration of $\overleftarrow{v}^P$ with respect to the final position : for all $y \in M$, we denote

$$\overleftarrow{v}^y = \overleftarrow{v}^{P(\cdot|X_1=y)} \tag{8.4.1}$$

There is a strong link between forward and backward stochastic velocity through time-reversed transformation. We denote by X^* the time-reversed coordinate process : $X^*_t = X_{1-t}$ for all $t \in [0,1]$. Let P^* be the time-reversed measure $X^*_{\#}P$. For all $t \in [0,1]$, we have :

$$\overleftarrow{v}^P_t = -\overrightarrow{v}^{P^*}_{1-t} \circ X^*. \tag{8.4.2}$$

For all $t \in [0,1]$, as $\overrightarrow{v}_t$ can be seen as functions of only $X_{[0,t]}$, $\overleftarrow{v}_t$ can be seen as a function of $X_{[t,1]}$.

Theorem 8.4.2. *For P_0 almost all $y \in M$, the backward stochastic velocity $\overleftarrow{v}^y$ satisfies :*

$$\begin{cases} \left(\partial_t + \nabla_{\overleftarrow{v}^y}\right) \overleftarrow{v}^y = \frac{a}{2}\Box \overleftarrow{v}^y - \mathbb{1}_{\mathcal{T}}(t)\nabla ap, & 0 \le t < 1, t \notin \mathcal{S}, z \in M, \\ \overleftarrow{v}^y_t - \overleftarrow{v}^y_{t^-} = \theta_t(.), & t \in S, z \in M, \\ \langle \overleftarrow{v}^y, \nu(z)\rangle = 0, & z \in \partial M, \\ \overleftarrow{v}^y_0 = -\nabla\eta(., y), & z \in M. \end{cases} \tag{8.4.3}$$

Furthermore, there exist a scalar potential φ^y satisfying a second order Hamilton-Jacobi equation, such that

$$\overleftarrow{v}^P_t (X) = -a\nabla\varphi_t^{X_1}(X_t), \; P\text{-}a.s. \tag{8.4.4}$$

Proof. The reference measure R is reversible : $R = R^*$. Then, the measure P^* has the form

$$P^* = \exp\left(\eta * (X_0, X_1) + \sum_{s\in\mathcal{S}*} \theta^*_s(X_s) + \int_{\mathcal{T}}^* p^*_t(X_t)\, dt\right) R, \tag{8.4.5}$$

where $\eta * (x, y) = \eta(y, x)$, $\mathcal{S}* = \{1 - s : s \in \mathcal{S}\}$, $\theta^*_s = \theta_{1-s}$, $\mathcal{T}* = \{1 - t : t \in \mathcal{T}\}$ and $p^*_t = p_{1-t}$. By analogy with (8.2.2), we define the following function φ^y by : for all $0 \le t \le 1$, for all $z \in M$,

$$\varphi^y_t(z) = \log \mathbb{E}_{R^x}\left[\exp\left(\eta^*(y, X_1) + \sum_{s\in\mathcal{S}^*, s>t} \theta^*_s(X_s) + \int_{\mathcal{T}^*\cap]t,1]} p^*_r(X_r)\, dr\right)\middle| X_t = z\right]. \tag{8.4.6}$$

According to Theorem 8.3.2, $a\varphi^y$ is a scalar potential of the forward stochastic velocity $\overrightarrow{v}^{y*} = \overrightarrow{v}^{P^*(\cdot|X_0=y)}$ and satisfies the system :

$$\begin{cases} \partial_t\varphi^y - \frac{a}{2}\Delta\varphi^y + a|\nabla\varphi^y|^2 + \mathbb{1}^*_{\mathcal{T}}(t)p^* = 0, & 0 \le t < 1, t \notin \mathcal{S}^*, z \in M, \\ \varphi^y(t, .) - \varphi^y(t^-, .) = -\theta^*_t(.), & t \in \mathcal{S}^*, \\ \langle\nabla\varphi^y(., z), \nu(z)\rangle = 0, & z \in \partial M, \\ \varphi^y(1, .) = \eta^*(x, .), & t = 1. \end{cases} \tag{8.4.7}$$

Using Equation (8.4.2), we have $\overleftarrow{v}^y_t (X) = -\overrightarrow{v}^{y*}_{1-t} (X_t)$ for all $0 \le t \le 1$. This ends the proof. $\square$

The system (8.4.3) is more satisfying. In its first equation, the Newton part, all the terms can be interpreted as in Navier-Stokes equation. Also, the impermeability condition is still fulfilled. However, the continuity equation seems to be out of range for $\overleftarrow{v}$. This is where the current velocity comes on.

8.5 Continuity equation

In this section, we try to obtain the last part of Navier-Stokes equation : the continuity equation. As the backward velocity does not seem to satisfy it, we need to introduced a new characteristic velocity: the current velocity. Firstly, we define the average forward and backward velocities $\overset{\hookrightarrow}{v}$ and $\overset{\hookleftarrow}{v}$. They are the mean of forward and backward velocity with respect to the initial and final position respectively :

$$\overset{\hookrightarrow}{v}_t(z) = \mathbb{E}_P\left[\overset{X_0\rightharpoonup}{v}_t|X_t = z\right] \quad \text{and} \quad \overset{\hookleftarrow}{v}_t(z) = \mathbb{E}_P\left[\overset{\leftharpoonup X_1}{v}_t|X_t = z\right]. \tag{8.5.1}$$

For $0 \le \alpha \le 1$, we define the α-velocity by

$$v_t^\alpha = (1-\alpha)\overset{\hookrightarrow}{v}_t + \alpha\overset{\hookleftarrow}{v}_t, \; \forall t \in [0,1]. \tag{8.5.2}$$

It is a convex mean of the average velocities. The current velocity, denoted as v_{cu} is the 1/2-velocity.

Theorem 8.5.1. *Assuming that $\mathcal{T} = [0,1]$, the current velocity v_{cu}^P satisfies the continuity equation*

$$\partial_t\mu_t + \operatorname{div}(\mu_t v_{cu}) = 0. \tag{8.5.3}$$

This equation has to be understood in distribution sense : for all $f \in \mathcal{C}_c^\infty(M)$,

$$\partial_t\mu_t(f) + \mu_t(\langle df, v_{cu}\rangle) = 0. \tag{8.5.4}$$

Proof. Let $f \in \mathcal{C}^\infty(M)$, and $0 \le t \le 1$. On one hand, we have :

$$\begin{aligned}
\mu_t(f) &= \mathbb{E}_P[f(X_t)] \\
&= \mathbb{E}_P\left[f(X_0) + \int_0^t \langle df, \overset{\rightharpoonup P}{v}_s\rangle\, ds + \frac{a}{2}\int_0^t \Delta f(X_s)\, ds + \int_0^t \langle df, \nu_{X_s}\rangle\, dL_s\right] \\
&= \mu_0(f) + \int_0^t \mu_s\left(\langle df, \overset{\hookrightarrow}{v}_s\rangle + \frac{a}{2}\Delta f\right) ds + \mathbb{E}_P\left[\int_0^t \langle df, \nu_{X_s}\rangle\, dL_s\right]
\end{aligned}$$

On the other hand, under the reversed law P^*, X is a semi-martingale with drift $\overset{\rightharpoonup P^*}{v} dt + \nu dL^*$ where the relation between $\overset{\leftharpoonup P}{v}$ and $\overset{\rightharpoonup P^*}{v}$ is given by (8.4.2) and $L_t(X) = L^*_{1-t}(X^*)$ for all $0 \le t \le 1$. We have :

$$\begin{aligned}
\mu_t(f) &= \mathbb{E}_{P^*}[f(X_{1-t})] \\
&= \mathbb{E}_{P^*}\left[f(X_0) + \int_0^{1-t} \langle df, \overset{\rightharpoonup P^*}{v}_s(X)\rangle\, ds + \frac{a}{2}\int_0^{1-t} \Delta f(X_s)\, ds + \int_0^{1-t} \langle df, \nu_{X_s}\rangle\, dL_s^*(X)\right]
\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}_{P^*}\left[f(X_0) + \int_t^1 \langle df, \overrightarrow{v}^{P^*}_{1-s}(X)\rangle\, ds + \frac{a}{2}\int_t^1 \Delta f(X^*_s)\, ds + \int_t^1 \langle df, \nu_{X_s}\rangle\, dL^*_{1-s}(X)\right] \\
&= \mathbb{E}_{P^*}\left[f(X_0) - \int_t^1 \langle df, \overleftarrow{v}^{P}_{s}(X^*)\rangle\, ds + \frac{a}{2}\int_t^1 \Delta f(X^*_s)\, ds + \int_t^1 \langle df, \nu_{X_s}\rangle\, dL_s(X^*)\right] \\
&= \mu_1(f) + \int_t^1 \mu_s\left(\langle df, -\overleftrightarrow{v}_s(X)\rangle + \frac{a}{2}\Delta f\right) ds + \mathbb{E}_P\left[\int_t^1 \langle df, \nu_{X_s}\rangle\, dL_s\right]
\end{aligned}
$$

Before differentiating, we need to show that the terms with local time are regular. For $\varepsilon > 0$, we denote $\partial^\varepsilon M$ the ε-tubular neighbourhood of ∂M. We have :

$$
\int_0^t \langle df, \nu_{X_s}\rangle\, dL_s = \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon}\int_0^t \langle df, \nu_{X_s}\rangle \mathbb{1}_{X_s\in\partial^\varepsilon M}\, ds. \tag{8.5.5}
$$

Then, we have :

$$
\begin{aligned}
\mathbb{E}_P\left[\int_0^t \langle df, \nu_{X_s}\rangle\, dL_s\right] &= \frac{1}{2}\int_0^t \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\mathbb{E}_P\left[\langle df, \nu_{X_s}\rangle \mathbb{1}_{X_s\in\partial^\varepsilon M}\right] ds \\
&= \frac{1}{2}\int_0^t \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\mu_s(\langle df, \nu\rangle \mathbb{1}_{\partial^\varepsilon M})\, ds \\
&= \frac{1}{2}\int_0^t \underline{\mu}_s(\langle df, \nu\rangle)
\end{aligned}
$$

where $\underline{\mu}$ denote the normalised surface measure associated to μ. It follows that for all $t \in [0,1]$,

$$
\partial_t\mu_t(f) = \mu_t(\langle \overrightarrow{v}_t, df\rangle + \frac{a}{2}\Delta f) + \underline{\mu}_t(\langle \nu, df\rangle) = \mu_t(\langle \overleftarrow{v}_t, df\rangle - \frac{a}{2}\Delta f) - \underline{\mu}_t(\langle \nu, df\rangle). \tag{8.5.6}
$$

This ends the proof. □

In the incompressible case of BS where $\mu_t = \mathrm{vol}$, the continuity equation becomes

$$
\mathrm{div}(v_{cu}) = 0. \tag{8.5.7}
$$

It is the incompressibility condition of the Navier-Stokes equations.

Chapter 9

Existence of solutions

We prove a satisfying criterion of existence of solution for the incompressible problem in several manifolds. We give a method to construct many more example with quotient. We explore a non-incompressible problem.

Contents

9.1 Introduction

In this chapter, we want to give examples of case where the BS problem admit a solution. In the spirit of Theorem 6.5.2, we know that if a measure can fulfil the entropy and marginal condition then there exists a unique solution but this does not prove the existence of solution : it is not a verifiable criterion. In [5], this idea was used to prove a condition of existence of solution for the incompressible problem on a torus $\mathbb{T}^n$. The highlighted criterion is the finiteness of entropy of the endpoints measure π. Their proof is not really specific to toruses except for one critical argument : is the law at a fixed time of a Brownian bridge, whose endpoints are uniformly and independently distributed,the volume measure? Translations in the torus allow to prove it with a straightforward calculation. The goals of this chapter is to find examples of manifold with boundary admitting solutions for the incompressible Brenier-Schrödinger problem and to investigate deeper the conditions of existence.

In Section 9.2, we extend the result of [5] ot torus to compact symmetric spaces. The method is similar but the more general structure force us to use less specific arguments. In Section 9.3, we prove haw symmetric spaces are a key to obtain many example on manifolds with boundary, or manifolds with corner, as parallelepiped of dimension n, or equilateral triangles. In Section 9.4 we end with a more exotic example. There, the space is not compact and the marginal constraint is not incompressible. It make arise an integrability condition hidden by the compactness.

9.2 Existence of a solution on symmetric spaces

In this section, we prove the existence of solution in compact symmetric spaces. The simplest examples of such space are toruses and spheres. A symmetric space cannot have boundary but we will see in the next section that it is the main ingredient to find examples of manifold with boundary. The idea of the proof is to exhibit a measure satisfying the entropy and marginals constraints.

Let (M, g) be a compact Riemannian manifold and G the group of isometries acting on M. We assume that the action of G is transitive. The main example of such spaces are the globally symmetric spaces. The renormalised Riemannian volume vol is a G-invariant probability measure on M. In this particular case, we can prove a result of existence of a solution to the incompressible Brenier-Schrödinger problem. The reference path measure R is the reversible Brownian motion on the manifold M and the marginal constraint is $\mu_t = \mathrm{vol}$, for all t. We denote by (H_M) the problem

$$H(P|R) \to \min; [P_t = \mathrm{vol}, \forall 0 \leq t \leq 1], P_{01} = \pi. \tag{IBS}$$

The idea of the proof is to find a path measure Q of finite relative entropy and satisfying the marginal conditions.The candidate for such a measure is

$$Q = \int_{M^3} R(.|X_0 = x, X_{1/2} = z, X_1 = y)\, \sigma(dxdzdy), \tag{9.2.1}$$

with $\sigma(dxdzdy) = \pi(dxdy)\,\mathrm{vol}(dz)$ in $P(M^3)$. It extends the result in [5] of existence on the torus, using the same property of invariance of the Brownian motion and the Riemannian volume, under isometries.

Proposition 9.2.1. *The path measure Q satisfies the marginal and endpoints constraints $[P_t = \mathrm{vol}, \forall 0 \leq t \leq 1]$ and $P_{01} = \pi$. In addition, if $H(\pi|R_{01}) < \infty$, then $H(Q|R) < \infty$.*

Proof. First, remark that, as R is a Markov measure, we have

$$R(.|X_0 = x, X_{1/2} = z, X_1 = y) = R(X_{[0,1/2]} \in .|X_0 = x, X_{1/2} = z)$$

$$\times R(X_{[1/2,1]} \in .|X_{1/2} = z, X_1 = y).$$

Now, let us check the endpoints constraint. Let A and B measurable subsets of M. We have :

$$\begin{aligned}
Q_{01}(A \times B) &= Q(X_0 \in A, X_1 \in B) \\
&= \int_{M^3} R(X_0 \in A|X_0 = x, X_{1/2} = z) R(X_1 \in B|X_{1/2} = z, X_1 = y)\, \sigma(dxdzdy) \\
&= \int_{M^3} \mathbb{1}_A(x)\mathbb{1}_B(y)\, \sigma(dxdzdy) \\
&= \sigma(A \times M \times B) \\
&= \pi(A \times B).
\end{aligned}$$

So $Q_{01} = \pi$.

Then, we prove that Q_t is G-invariant for all t. Let $0 \leq t \leq 1/2$ and f a bounded measurable function on M. We have

$$\begin{aligned}
\int_M f\, dQ_t &= \int_{M^3} \mathbb{E}_R \left[f(X_t)|X_0 = x, X_{1/2} = z \right] \sigma(dxdzdy) \\
&= \int_{M^2} \mathbb{E}_R \left[f(X_t)|X_0 = x, X_{1/2} = z \right] \mathrm{vol}(dx)\, \mathrm{vol}(dz).
\end{aligned}$$

For all isometry $g \in G$, using the invariance in law of the Brownian motion and the invariance of the Riemannian volume under isometry, we have :

$$\begin{aligned}
\int_M f \circ g\, dQ_t &= \int_{M^2} \mathbb{E}_R \left[f \circ g(X_t)|X_0 = x, X_{1/2} = z \right] \mathrm{vol}(dx)\, \mathrm{vol}(dz) \\
&= \int_{M^2} \mathbb{E}_R \left[f(X_t)|X_0 = g(x), X_{1/2} = g(z) \right] \mathrm{vol}(dx)\, \mathrm{vol}(dz) \\
&= \int_{M^2} \mathbb{E}_R \left[f(X_t)|X_0 = x, X_{1/2} = z \right] \mathrm{vol}(dx)\, \mathrm{vol}(dz) \\
&= \int_M f\, dQ_t.
\end{aligned}$$

So Q_t is a G-invariant probability on M. According to [40], vol is the only G-invariant Radon probability measure (see Proposition 476C). So, $Q_t = \mathrm{vol}$, $\forall 0 \leq t \leq 1/2$. The result is obtained, mutatis mutandis, for $1/2 \leq t \leq 1$. Then, Q satisfy the marginal constraint.

We finish by the entropy to prove the finiteness criterion. We denote

$$Q_{0,1/2,1} = Q(X_0 \in ., X_{1/2} \in ., X_1 \in .) \tag{9.2.2}$$

and $Q^{xzy} = Q(.|X_0 = x, X_{1/2} = z, X_1 = y)$. We have

$$\begin{aligned} H(Q|R) &= H(Q_{0,1/2,1}|R_{0,1/2,1}) + \int_{M^3} H(Q^{xzy}|R^{xzy})\, Q_{0,1/2,1}(dxdzdy) \\ &= H(\sigma|R_{0,1/2,1}) \\ &= H(\sigma_{01}|R_{01}) + \int_{M^3} H(\sigma_{xy}|R^{xy}_{1/2}\, \sigma_{01}(dxdy) \\ &= H(\pi|R_{01}) + \int_{M^3} H(\mathrm{vol}\,|R^{xy}_{1/2}\, \pi(dxdy) \end{aligned}$$

By definition of the relative entropy, we have

$$H(\mathrm{vol}\,|R^{xy}_{1/2}) = \int_M \log\left(\frac{d\,\mathrm{vol}}{dR^{xy}_{1/2}}\right) \mathrm{vol}(dz). \tag{9.2.3}$$

We denote p the heat kernel on M. We have

$$\frac{dR^{xy}_{1/2}}{d\,\mathrm{vol}}(z) = \frac{p_{1/2}(x,z)p_{1/2}(z,y)}{p_1(x,y)}. \tag{9.2.4}$$

This quantity is continuous in x, y and z. As M is compact, the density can be bounded uniformly in the three variables. So the relative entropy $H(Q|R)$ is finite if and only if $H(\pi|R_{01})$ is finite. $\square$

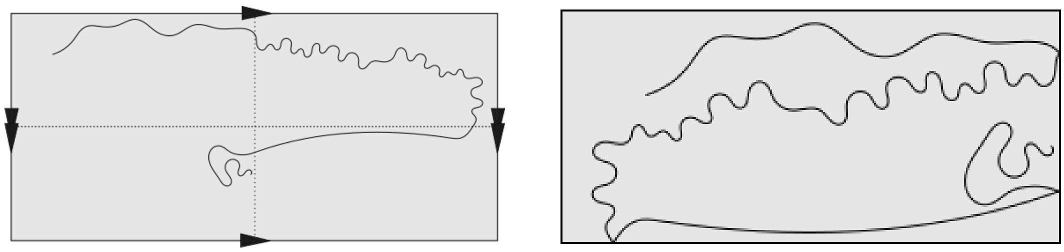
As we can see, the only argument using any property of M is the proof that the law, at a fixed time, of a Brownian bridge between to independent uniformly distributed random variables, is the uniform measure vol. It seems hat on any space satisfying that, we can obtain the same result of existence by the same argument. It would be interesting to understand what is occurring in a space without this property.

As a corollary of 6.5.2 and Proposition 9.2.1, we have the following criterion of existence.

Corollary 9.2.2. *The Brenier-Schrödinger problem IBS admits a unique solution if and only if $H(\pi|R_{01}) < \infty$.*

9.3 Reflected Brownian motion in quotient space

We shall describe here a relation between the Brenier-Schrödinger problem on compact Riemannian manifolds and on some quotients of these. For instance, we want to see the n-hypercube as a quotient of the torus $\mathbb{T}^n$ (see Figure 9.1) or a curved n-ball as a quotient of the sphere $\mathbb{S}^n$. In particular, we will allow singularities for the boundaries and talk about boundaries with corners.



(a) (b)

Figure 9.1: A path in the torus (a) and its projection in the rectangle (b)

Definition 9.3.1 (Manifold with corners)**.** *Let N be a Hausdorff second countable topological space and let $n > 0$ be a positive integer. Suppose that we have a family $\{\varphi_\lambda\}_{\lambda\in\Lambda}$ of homeomorphisms*

$$\varphi_\lambda : U_\lambda \subset N \to V_\lambda \subset [0,\infty)^n$$

where U_λ (respectively V_λ) is an open subset of N (respectively of $[0,\infty)^n$). We say that $\{\varphi_\lambda\}_{\lambda\in\Lambda}$ is a smooth atlas with corners *if*

$$\bigcup_{\lambda\in\Lambda} U_\lambda = N$$

and, for every $\mu, \nu \in \Lambda$,

$$\varphi_\mu \circ \varphi_\nu^{-1} : \varphi_\nu\left(U_\mu \cap U_\nu\right) \to \varphi_\mu\left(U_\mu \cap U_\nu\right)$$

has a smooth extension to an open subset of $\mathbb{R}^n$. We will refer to $(N, \{\varphi_\lambda\}_{\lambda\in\Lambda})$ as a manifold with corners[1].

It will be useful to have in mind triangles (and squares) as the prototypical examples, the smooth atlas being given by the set of all diffeomorphisms from an open subset of the triangle (or square) to the open subsets of $[0,\infty)^n$. The notions of tangent bundle, Riemannian metric, Levi-Civita connection and stochastic differential equations can be carried over to manifolds with corners.

We fix some notations about the boundary ∂N. The points that correspond to the singular points of the boundary of $[0,\infty)^n$ will be called the *corner points.* A boundary point x that is not a corner point will be called a *regular boundary point* and there is a well-defined unit inward-pointing normal vector $\nu_x \in T_xN$ at x.

We will be interested on the reflected Brownian motion on these manifolds. As for the case of manifolds with boundary, for every $x \in M$ we consider $\sigma_x \in L(\mathbb{R}^m, T_xM)$ such that $\sigma_x\sigma_x^* = \mathrm{id}_{T_xM}$ and such that the family $\{\sigma_x\}_{x\in M}$ is smooth, i.e. the map $\sigma : M \times \mathbb{R}^m \to TM$ defined by $\sigma(x,w) = \sigma_x(w)$ is smooth.

[1]The standard definition considers a maximal smooth atlas with corners $\{\varphi_\lambda\}_{\lambda\in\Lambda}$. For simplicity of exposition we consider any smooth atlas in the definition.

Proposition 9.3.2 (Reflected Brownian motion)**.** *There exists a continuous stochastic process $(X_t)_{t\in[0,1]}$ on N that, almost surely, does not touch the corner points and such that*

$$dX_t = \sigma_{X_t} dW_t + \nu_{X_t} dL_t,$$

where W is a Brownian motion on $\mathbb{R}^m$ and L is a non-decreasing process such that

$$\int_0^1 \mathbb{1}_{\mathring{N}}(X_s) dL_s = 0.$$

Moreover, the law of $(X_t)_{t\in[0,1]}$ solely depends on the law of X_0 and the metric of N. The process $(X_t)_{t\in[0,1]}$ will be called a reflected Brownian motion on N.

There is another characterization of the reflected Brownian motion that explicitly shows how it depends only on the metric.

Proposition 9.3.3 (Reflected Brownian motion)**.** *Let $(X_t)_{t\in[0,1]}$ be a reflected Brownian motion on N. For every smooth function $f : N \to \mathbb{R}$ such that $df_x \cdot \nu_x = 0$ at every regular boundary point x, we have that*

$$f(X_t) - \int_0^t \Delta f(X_s) ds$$

is a martingale with respect to the filtration composed by $\mathcal{F}_t = \sigma((X_s)_{0\leq s\leq t})$. Moreover, the law of $(X_t)_{t\in[0,1]}$ is characterized by this fact and the law of X_0.

Now we will describe how to obtain nice quotient spaces. Our setting will be the following. Suppose that M is a connected compact Riemannian manifold (without boundary) and that G is a finite group of isometries of M. For the quotient M/G to be a manifold with corners we will consider a particular class of groups of isometries. For $x \in M$, consider the group

$$G_x = \{g \in G : \, g(x) = x\},$$

and the induced subgroup of $O(T_xM)$,

$$\mathbb{G}_x = \{dg_x \in O(T_xM) : \, g \in G_x\}.$$

Let R_x be the set of reflections in $\mathbb{G}_x$, i.e. $T \in \mathbb{G}_x$ belongs to R_x if and only if

$$\{u \in T_xM : \, Tu = u\} \subset T_xM \text{ has codimension one.}$$

Definition 9.3.4 (Reflection group)**.** *We shall say that G is a reflection group (of isometries) if*

$$\mathbb{G}_x \textit{ is the group generated by } R_x$$

for every $x \in M$.

We will be interested in the set $N = M/G$ which has a topological structure induced by the quotient map $q : M \to N$. Suppose that G is a reflection group. We shall make of N a manifold with corners in the following way. Let $y \in N$ and take any $x \in M$ with $q(x) = y$.

If $G_x = \{e\}$ then there exists an open neighbourhood V of x such that $gV \cap hV = \emptyset$ for every $g \neq h$ in G. Since q is open and $q|_V$ is injective and continuous we have that $q|_V : V \to q(V)$ is an homeomorphism and we can assume that (by taking a smaller V if necessary) V is diffeomorphic to an open subset of $(0, \infty)^n$. This gives an atlas to the open set of points that can be written as $q(x)$ with $G_x = \{e\}$. We can even define a metric on this open set with the help of these $q|_V$.

If $G_x \neq \{e\}$ we consider the exponential map

$$\exp_x : W \subset T_xM \to V \subset M$$

on an open neighbourhood W of 0 invariant under $\mathbb{G}_x$ such that $\exp_x|_W$ is a diffeomorphism onto its image V. Moreover, by choosing V small enough we will assume that $gV \cap V = \emptyset$ for every $g \notin G_x$. Since

$$g \exp_x(w) = \exp_x(dg_x w) \tag{9.3.1}$$

for $g \in G_x$ and $w \in T_xM$, the open set V is invariant under G_x. Equation (9.3.1) tells us that the action of G_x on W (as $\mathbb{G}_x$) is isomorphic to the action of G_x on V. Then, we only need to understand

$$W/\mathbb{G}_x.$$

But, since $\mathbb{G}_x$ is a reflection group, we know that $T_xM/\mathbb{G}_x$ can be identified with a particular fundamental domain of the action of $\mathbb{G}_x$ on T_xM, called closed chamber (see [48, Section 1.12]), and, in particular, it has a structure of a manifold with corners so that $W/\mathbb{G}_x$ inherits this structure. Using $\exp_x$ we have given to the open set $V/G_x \simeq q(V)$ the structure of a manifold with corners. In fact, if $C \subset T_xM$ is a closed chamber, we have identified $\exp_x(C \cap W)$ with $q(V)$. The latter identification gives a Riemannian metric to $q(V)$.

We have shown the following lemma.

Lemma 9.3.5. *Suppose that G is a reflection group of isometries of M. Then,*

N has a (unique) structure of a Riemannian manifold with corners

such that, for every $x \in M$, there exists a neighbourhood $U \subset N$ of $q(x)$ and an isometric submersion $s : U \to M$ that is a local inverse of q, i.e. such that

$$q \circ s(x) = x \text{ for every } x \in U.$$

Proof. The smooth structure around a point x is induced by the quotient of T_xM by $\mathbb{G}_x$. It is, locally, a chamber for the reflection group $\mathbb{G}_x$. $\square$

Lemma 9.3.6. *Let G be a finite group of isometries of M. Then, there exists an open subset V of M such that*

- *$gV \cap hV = \emptyset$ for every $g \neq h$ in G and*
- $\sigma\left(M \Big\backslash \bigcup_{g\in G} gV\right) = 0$.

Proof. Let $x \in M$ such $G_x = \{e\}$ and define the set

$$V = \{y \in M : \forall g \in G \setminus \{e\},\, d(x, y) < d(gx, y)\}\,,$$

where d is the distance function on the Riemannian manifold M. Since G is a group of isometries we have that

$$hV = \{y \in M : \forall g \in G \setminus \{h\},\, d(hx, y) < d(gx, y)\}\,.$$

We only need to see that, for $\xi \neq \zeta$ in M, the set

$$\{y \in M : d(\xi, y) = d(\zeta, y)\}$$

has σ-measure zero. This is true since the map $y \mapsto d(\xi, y) - d(\zeta, y)$ is smooth and regular outside the cut locus of ξ and ζ and since every cut locus has σ-measure zero. $\square$

Notice that, in particular, for every $g \in G$ and $x \in gV$ the group G_x contains only the identity so that $q|_{gV}$ is an isometry onto its image. There is an intuitive relation between a Brownian motion on M an on its quotient by a reflection group.

Lemma 9.3.7. *Suppose that G is a reflection group of isometries of M. Let $\{B_t^x\}_{t\geq 0}$ be a Brownian motion on M starting at $x \in M$. Then $\{q(B_t^x)\}_{t\geq 0}$ does not touch the corner points almost surely and*

$$\{q(B_t^x)\}_{t\geq 0} \textit{ is a reflected Brownian motion on } M/G.$$

Proof. This can be seen by using the second characterization of a reflected Brownian motion. Let $f : N \to \mathbb{R}$ be a smooth map such that $df_x\nu_x = 0$ at every regular boundary point x and consider the function $F = f \circ q$ which is a smooth function on $\{x \in M : G_x = \{e\}\}$ and it is differentiable at the points $x \in M$ such that $q(x)$ is a regular boundary point. If F happens to be regular enough we know that

$$F(X_t) - \int_0^t \Delta F(X_s)ds$$

is a martingale with respect to the filtration induced by $(X_s)_{0\le s\le t}$. By using that

$$\Delta F = (\Delta f) \circ q$$

we have proved that

$$f(q(X_t)) - \int_0^t \Delta f(q(X_s))ds$$

is a martingale with respect to the filtration given by $\mathcal{G}_t = \sigma((X_s)_{0\le s\le t})$. In particular, since it is adapted to the filtration given by $\mathcal{F}_t = \sigma(\{q(X_s)\}_{0\le s\le t})$ and since $\mathcal{F}_t \subset \mathcal{G}_t$, it is also a martingale with respect to this filtration. Finally, since the subset of functions $f : N \to \mathbb{R}$ such that $F = f \circ q$ is regular and whose normal derivative is zero is dense (in the uniform topology for them and their Laplacian), we have proved the assertion. $\square$

Now, denote the normalized volume measure on $N = M/G$ by $\tilde{\sigma}$. Let R be the law of the Brownian motion on M whose initial position has law σ and let $\tilde{R}$ be the law of the reflected Brownian motion on N whose initial position has law $\tilde{\sigma}$. We have the following result.

Lemma 9.3.8. *Denote by $q(R)$ the image measure of R by the map induced by q on $C([0,1], N)$. Then, $q(R) = \tilde{R}$.*

Proof. By Lemma 9.3.7, $q(R)$ is the law of the Brownian motion on N whose initial position is distributed according to $q_*\sigma$, the image measure of σ by q. It is enough, then, to notice that $q_*\sigma = \tilde{\sigma}$.

$$q_*\sigma = q_*\left(\sum_{g\in G} \sigma|_{gU}\right) = \sum_{g\in G} q_*\left(\sigma|_{gU}\right).$$

We have that

$$\tilde{\sigma}\left(q(M \setminus \cup_{g\in G}\, gU)\right) = 0. \tag{9.3.2}$$

since the measure of ∂N is zero and, on the complement of $q^{-1}(\partial N)$, the map q is smooth so that the image of a set of measure zero has also measure zero. Since $q|_{gU}$ is an isometry onto its image we have that

$$q_*\left(\sigma|_{gU}\right) = \text{ volume measure on } N,$$

where we have used (9.3.2) which says that $\tilde{\sigma}\left(N \setminus q(gU)\right) = 0$. We obtain

$$q_*\sigma = \mathrm{card}(G)\,(\text{volume measure on } N)$$

which, after normalizing, concludes the proof. $\square$

Theorem 9.3.9. *Let π be a probability measure on $M \times M$ with both marginals equal to σ and such that $H(\pi|R_{01}) < \infty$. If $H_{M,\pi}$ admits a solution, then $H_{N,(q\times q)_*\pi}$ admits a solution. In particular, if $H_{M,\pi}$ admits a solution for every such π, then $H_{N,\tilde{\pi}}$ admits a solution for every probability measure $\tilde{\pi}$ on $N \times N$ with both marginals equal to $\tilde{\sigma}$ and such that $H(\tilde{\pi}|\tilde{R}_{01}) < \infty$.*

Proof. Let Q be a probability measure on $C([0,1], M)$ such that $Q_{01} = \pi$, $Q_t = \sigma$ for every $t \in [0,1]$ and $H(Q|R) < \infty$. We need to find a probability measure $\tilde{Q}$ on $C([0,1], \tilde{M})$ such that $\tilde{Q}_{01} = \pi$, $\tilde{Q}_t = \tilde{\sigma}$ for every $t \in [0,1]$ and $H(\tilde{Q}|\tilde{R}) < \infty$. Notice that

$$H(q(Q)|q(R)) \leq H(Q|R) < \infty.$$

Since $q(Q)$ satisfies the marginal assumptions and since $q(R) = \tilde{R}$, the proof is concluded by taking $\tilde{Q} = q(Q)$. Now, to prove the second assertion we need to write $\tilde{\pi}$ as $q_*\pi$ for some nice π. For this, we shall use Lemma 9.3.6. Since $H(\tilde{\pi}|\tilde{R}_{01}) < \infty$ we have that $H(\tilde{\pi}|\tilde{\sigma} \otimes \tilde{\sigma}) < \infty$. In particular, $\tilde{\pi}$ gives measure zero to $M \times M \setminus q(U) \times q(U)$. For every $(g,h) \in G \times G$, consider the map

$$(q|_{gU} \times q|_{hU})^{-1} : q(U) \times q(U) \to gU \times hU$$

and consider the measure

$$\pi_{g,h} = (q|_{gU} \times q|_{hU})_*^{-1}\, \tilde{\pi}$$

which satisfies

$$(q \times q)_* \pi_{g,h} = \tilde{\pi},$$

Nevertheless, it does not satisfy the marginal conditions. Notice that, if

$$\alpha_{g,h} = (q|_{gU} \times q|_{hU})_*^{-1}\, (\tilde{\sigma} \times \tilde{\sigma}) \quad \text{and} \quad \sigma_g = (q|_{gU})_*^{-1}\, \tilde{\sigma}$$

then

$$\alpha_{g,h} = |G|^2\, (\sigma \times \sigma)\, |_{gU \times hU} \quad \text{and} \quad \sigma_g = |G|\, \sigma|_{gU}.$$

Moreover, the first marginal of $\pi_{g,h}$ is σ_g and its second marginal is σ_h. Then, if we define

$$\pi = \frac{1}{|G|^2} \sum_{(g,h) \in G \times G} \pi_{g,h},$$

we may notice that the first and second marginals of π are σ and that

$$(q \times q)_* \pi = \tilde{\pi}.$$

We can also find its entropy by integrating and obtain that

$$H(\pi|\sigma \otimes \sigma) = H(\tilde{\pi}|\tilde{\sigma} \otimes \tilde{\sigma}).$$

Since $H(\pi|\sigma \otimes \sigma) < \infty$ if and only if $H(\pi|R_{01}) < \infty$ and $H(\tilde{\pi}|\tilde{\sigma} \otimes \tilde{\sigma}) < \infty$ if and only if $H(\tilde{\pi}|\tilde{R}_{01}) < \infty$ we may conclude. □

We shall consider some simple examples of quotient spaces where Theorem 9.3.9 holds. Almost all of these will be quotients of the flat two-dimensional torus which we define now. Let u and v be two independent vectors of $\mathbb{R}^2$. We will denote by $\mathbb{T}_{u,v}$ the manifold

$$\mathbb{T}_{u,v} = \mathbb{R}^2/\{au + bv : a, b \in \mathbb{Z}\}$$

endowed with the Riemannian metric induced by $\mathbb{R}^2$. We begin by describing two examples that are actual two-dimensional manifolds with boundary (without corners).

Example 9.3.10 (Cylinder)**.** *Suppose that u and v are orthogonal. The map*

$$\begin{gathered}\{\alpha u + \beta v : (\alpha, \beta) \in [0,1]^2\} \to \{\alpha u + \beta v : (\alpha, \beta) \in [0,1]^2\} \\ xu + yv \mapsto xu + (1-y)v\end{gathered}$$

induces an isometry of $\mathbb{T}_{u,v}$ and the quotient space is isometric to the cylinder

$$\{z \in \mathbb{C} :\ 2\pi|z| = |u|\} \times [0, |v|/2].$$

Example 9.3.11 (Flat Möbius strip)**.** *Suppose that $|u| = |v|$. The map*

$$\begin{gathered}\{\alpha u + \beta v : (\alpha, \beta) \in [0,1]^2\} \to \{\alpha u + \beta v : (\alpha, \beta) \in [0,1]^2\} \\ xu + yv \mapsto yu + xv\end{gathered}$$

induces an isometry of $\mathbb{T}_{u,v}$ and the quotient space is isometric to the flat Möbius strip

$$[0, \|u+v\|/2] \times [0, \|u-v\|/2]/ \sim$$

where $\sim$ is the identification of the vertical sides in opposite directions.

Figure 9.2 shows a representation of the torus and the considered isometry is the reflection along the dotted diagonal. Figure 9.3 shows the canonical representation of the flat Möbius strip as part of (four times) the representation of the torus.

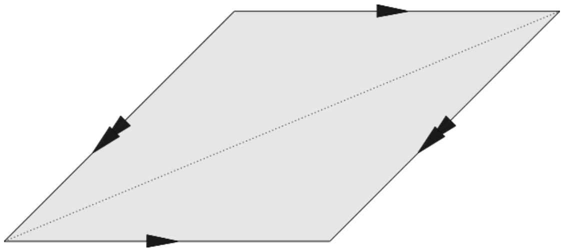


Figure 9.2: The Möbius strip as a quotient.

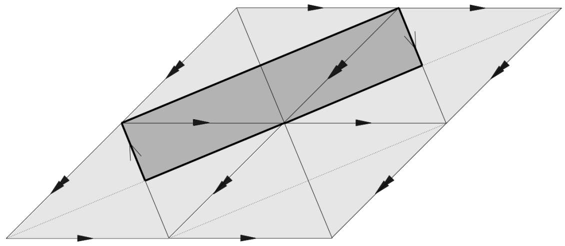


Figure 9.3: The Möbius strip.

The next four examples are two-dimensional manifolds with corners.

Example 9.3.12 (Rectangle)**.** *Suppose that u and v are orthogonal. The maps*

$$\begin{aligned}\{\alpha u + \beta v : (\alpha, \beta) \in [0,1]^2\} &\to \{\alpha u + \beta v : (\alpha, \beta) \in [0,1]^2\} \\ xu + yv &\mapsto xu + (1-y)v\end{aligned}$$

and

$$\begin{aligned}\{\alpha u + \beta v : (\alpha, \beta) \in [0,1]^2\} &\to \{\alpha u + \beta v : (\alpha, \beta) \in [0,1]^2\} \\ xu + yv &\mapsto (1-x)u + yv\end{aligned}$$

generate a reflection group of isometries of $\mathbb{T}_{u,v}$ and the quotient space is isometric to

$$[0, |u|/2] \times [0, |v|/2]$$

.

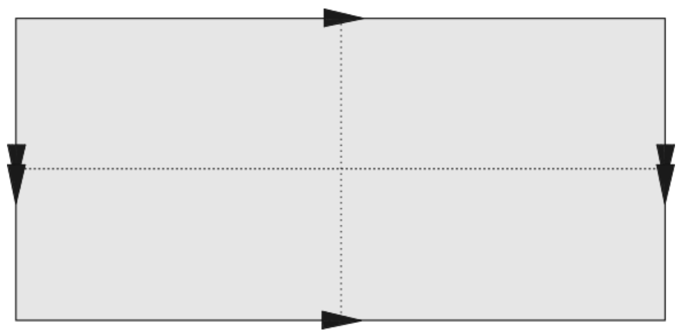


Figure 9.4: A rectangle as a quotient of the torus.

Example 9.3.13 (Isosceles right triangle)**.** *A* $45°$ *right triangle can be seen as a quotient of a square by a reflection along its diagonal. Using the previous example, we can also see it as a quotient of a torus (see Figure 9.5).*

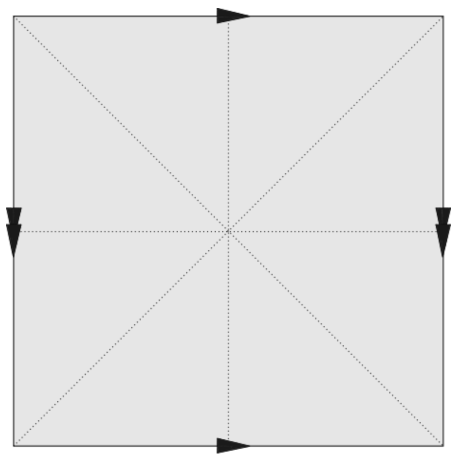


Figure 9.5: A 45° triangle rectangle as a quotient.

Example 9.3.14 (Equilateral triangle)**.** *If* $2u \cdot v = \|u\|\|v\|$, *the torus* $\mathbb{T}_{u,v}$ *can be seen as a quotient of an hexagon identifying opposite sides as in Figure 9.6. Then, if we consider the group generated by the reflections along the dotted lines in Figure 9.6 we can obtain an equilateral triangle as a quotient space.*

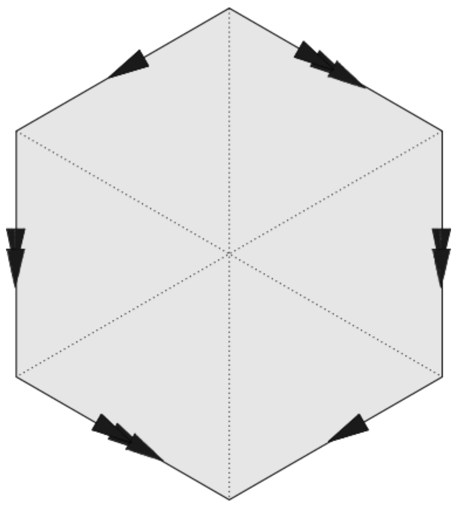


Figure 9.6: An equilateral triangle as a quotient of the torus.

Example 9.3.15 (60° right triangle)**.** *A* 60° *right triangle can be seen as a quotient of the equilateral triangle by a reflection. Using the previous example we can see it also as a quotient of a torus.*

Finally, as n-dimensional cases we consider the following examples.

Example 9.3.16 (n-hyperrectangle)**.** *Let $a_1, \dots, a_n > 0$ and let $u_1, \dots, u_n$ be orthogonal vectors in $\mathbb{R}^n$ such that $\|u_i\| = a_i$ for any $i \in \{1, \dots, n\}$. We may consider the flat n-dimensional torus*

$$\mathbb{T}^n = \mathbb{R}^n / \{m_1 u_1 + \cdots + m_n u_n : m_1, \dots, m_n \in \mathbb{Z}\}$$

and the group generated by the reflections induced by the family (indexed by $i \in \{1, \dots, n\}$) of maps

$$\begin{aligned} \{\alpha_1 u_1 + \cdots + \alpha_n u_n : \alpha_i \in [0,1]\} &\to \{\alpha_1 u_1 + \cdots + \alpha_n u_n : \alpha_i \in [0,1]\} \\ \sum_{j=1}^n x_j u_j &\mapsto \sum_{j \neq i} x_j u_j + (1 - x_i) u_i \end{aligned}$$

The quotient of $\mathbb{T}^n$ by this group is a $n-$hyperrectangle with lengths $a_1/2, \dots, a_n/2$.

Example 9.3.17 (Curved n-ball)**.** *Consider the n-dimensional sphere*

$$\mathbb{S}^n = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^n : |x_1|^2 + \cdots + |x_{n+1}|^2 = 1\}.$$

The quotient of $\mathbb{S}^n$ by the map

$$\begin{aligned} \mathbb{S}^n &\to \mathbb{S}^n \\ (x_1, \dots, x_n, x_{n+1}) &\mapsto (x_1, \dots, x_n, -x_{n+1}) \end{aligned}$$

is a curved n-ball.

9.4 The Gaussian case

In this section, we finish with a more exotic example. Actually, dealing with volume measure as marginal constraint implies working in compact spaces. All the known criteria of existence need compactness. As seen in the proofs, this assumption is used several times, and not only to say that *vol* is a finite measure. Here, we propose an other model where the compactness is, somehow, substituted by the finiteness of second order momentum of the constraint measure. Let Ω the space of paths from $[0,1]$ to $M = \mathbb{R}^n$. The reference measure R is still the reversible Brownian motion. We are looking to the following problem :

$$H(P|R) \to \min; [P_t = \mathcal{N}(0, 1/4\,\mathrm{id}), \forall 0 \le t \le 1], P_{01} = \pi, , \qquad (BS_\gamma)$$

where π is a probability on M^2. We consider the following path measure

$$Q = \int_{M^3} R(.|X_0 = x, X_{1/2} = z, X_1 = y)\, \pi(dxdy) \gamma_{1/4}(dz), \qquad (9.4.1)$$

where γ_{σ^2} denotes the density of $\mathcal{N}(0, \sigma^2\,\mathrm{id})$. This measure is build in the same spirit as (9.2.1).

Proposition 9.4.1. *The measure Q satisfies the marginal conditions $Q_{01} = \pi$ and $Q_t = \mathcal{N}(0, 1/4\,\mathrm{id}), \forall 0 \le t \le 1$. If $H(\pi|R_{01}) < \infty$ then $H(Q|R) < \infty$.*

Proof. The steps and arguments of the proof are the same as in the previous section. Firstly, as in the proof or Proposition 9.2.1, the endpoints condition $Q_{01} = \pi$ is obviously satisfied. Then, for $0 \le t \le 1/2$, we have

$$Q_t = \int_{M^2} R_t(.|X_0 = x, X_{1/2} = z)\, \gamma_{1/4}(dx)\gamma_{1/4}(dz) \tag{9.4.2}$$

where $R_t(.|X_0 = x, X_{1/2} = z)$ is the law, at time t of a Brownian bridge on $[0, 1/2]$ between x and z. It is a normal distribution $\mathcal{N}\left((1-2t)x + 2tz, t(1-2t)\right)$. So Q_t is a normal distribution and we have :

$$(1-2t)Y + 2tZ + \sqrt{t(1-2t)}W \sim Q_t, \tag{9.4.3}$$

where Y,$Z \sim \mathcal{N}(0, 1/4\,\mathrm{id})$ and $W \sim \mathcal{N}(0, \mathrm{id})$ are independent random variables. It follows that $Q_t = \mathcal{N}(0, 1/4\,\mathrm{id})$ for all $0 \le t \le 1/2$ and for all $0 \le t \le 1$ with the same argument.

It remain to verify the entropy condition. As in the symmetric space case, we have :

$$H(Q|R) = H(\pi|R_{01}) + \int_M H(\gamma_{1/4}|R^{xy}_{1/2})\, \pi(dxdy). \tag{9.4.4}$$

Using the heat kernel in M, we have :

$$\frac{dR^{xy}_{1/2}}{d\gamma_{1/4}}(z) = e^{2\langle z, x+y\rangle - \frac{1}{2}|x-y|^2}. \tag{9.4.5}$$

And then, the entropy is

$$H(\gamma_{1/4}|R^{xy}_{1/2}) = \frac{1}{2}|x-y|^2 \tag{9.4.6}$$

So we have

$$\begin{aligned} H(Q|R) &\le H(\pi|R_{01}) + \int_{M^2} \frac{1}{2}|x-y|^2\, \pi(dxdy) \\ &\le H(\pi|R_{01}) + \int_{M^2} (x^2 + y^2)\, \pi(dxdy) \\ &\le H(\pi|R_{01}) + 2\int_M x^2\, \gamma_{1/4}(dx) \\ &\le H(\pi|R_{01}) + \frac{n}{2}. \end{aligned}$$

This ends the proof. □

Corollary 9.4.2. *The Brenier-Schrödinger problem BS_γ admits a unique solution if and only if $H(\pi|R_{01}) < \infty$.*

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