Stochastic calculus on manifold and application to functional inequalities

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Framework

- M Riemannian manifold (smooth, complete);
- $V : M \to \mathbb{R}$ a potential (smooth);
- $\mu(dx) = e^{-V(x)}dx$ a probability measure (up to renormalisation).

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Goals

- Obtain functional inequalities;
- Understand the underlying stochastic processes.

Poincaré :

$$\mathrm{Var}_\mu(f) \leq \frac{1}{\rho} \int_{\mathbb{R}^n} |df|^2 \, d\mu.$$

Our tool : intertwining of semi-groups :
$\mathbf{P}$ semi-group on functions, $\mathbf{Q}$ semi-group on 1-forms
$\mathbf{P}$ and $\mathbf{Q}$ are intertwined by the differential d if for all $f \in \mathcal{C}_c^\infty(M)$,

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Assumptions : ergodicity i.e $\mathbf{P}_t f \to \mu(f)$ a.s.

$$\mathrm{Cov}_\mu(f,g) = \int_0^{+\infty} \int_M \langle d\mathbf{P}_t(f), dg \rangle \, d\mu \, dt.$$

Questions

- When is this true?
- How can we extend it?

Ground floor - Generators

Generator on $\mathcal{C}^\infty(M)$:

$$L = \Delta - \nabla V = \sum_i \frac{\partial^2}{\partial x_i^2} - \sum_i \frac{\partial V}{\partial x_i}\frac{\partial}{\partial x_i}.$$

L is symmetric with respect to μ :

$$\int L(f)g\, d\mu = -\int \langle df, dg\rangle\, d\mu = \int fL(g)\, d\mu.$$

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Intertwining

$$dLf = L^W df.$$

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$$dLf = L^W df.$$

L^W differential operator on 1-forms. Symmetric, non-positive.

$$L^W = L^{/\!/} - \mathcal{M},$$

- $L^{/\!/}$ symmetric non-positive
- "matrix" potential $\mathcal{M} = \mathrm{Ric} + \nabla^2 V$.

Basement - Stochastic processes

X^x : diffusion with generator L, starting from x

$$f(X_t) \stackrel{(m)}{=} f(x) + \int_0^t Lf(X_s)ds.$$

Example : $L = \Delta$ for Brownian motion

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Diffusion on TM, above X^x

$$DW_t = -\mathcal{M}W_t dt.$$

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Intertwining

W is the spacial derivative of an appropriate flow X :

$$W_t = \nabla X_t.$$

Upper floor - Semi-groups

L generates a $\mathcal{C}^0$-semi-group

$$\mathbf{P}_t f(x) = \mathbb{E}[f(X_t^x)\mathbb{1}_{t<\tau_x}].$$

Intertwining?

More assumptions are needed... even for the definition of the $\mathcal{C}^0$-semi-group associated to L^W (bounded forms are NOT bounded).

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L^W generates a L^2-semi-groups $\mathbf{Q}$.

Intertwining?

More assumptions are still needed...

An assumption on the potential $\mathcal{M}$:

Definition (Bakry-Émery criterion)

We assume that $\mathcal{M} = \mathrm{Ric} + \nabla^2 V$ is uniformly bounded from below :

$$\rho = \inf_{x \in M} \{\text{smallest eigenvalue of } \mathcal{M}\} > -\infty.$$

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Proposition

Under (BE), for all $x \in M$, for all $v \in T_x M$, for all $t \geq 0$, we have :

$$|W_t(v)| \leq e^{-\rho t} |v| \; a.s$$

$\longrightarrow$ $\mathcal{C}^0$-semi-group $\mathbf{Q}_t$ for continuous bounded 1-forms α :

$$\mathbf{Q}_t(\alpha) = \mathbb{E}\left[\langle \alpha, W_t . \rangle \mathbb{1}_{t<\tau}\right],$$

$\longrightarrow$ Commutation formula :

$$d\mathbf{P}_t(f) = \mathbf{Q}_t(df).$$

Theorem

If $\rho > 0$, then for all $f, g \in \mathcal{C}_c^\infty(M)$, we have:

$$\mathrm{Cov}_\mu(f, g) \leq \frac{1}{\rho} \|df\|_\infty \|dg\|_1.$$

In $\mathbb{R}^n$, $V(x) = |x|^4/4$. $\mathcal{M} = 2x.x^t + |x|^2 \,\mathrm{id} \geq 0$. But not $\geq k\,\mathrm{id} > 0$!

problem

The intertwining by d does not lead to Poincaré inequality.

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solution ?

Replace d by $(B^*)^{-1}d$, with $B : x \in M \to B_x \in \mathrm{GL}(T_x M)$, smooth.

Metric : $\langle \cdot, \cdot \rangle_B = \langle B^{-1} \cdot, B^{-1} \cdot \rangle$.

$$\mathrm{Cov}_\mu(f, g) = \int_0^{+\infty} \int_M \langle (B^*)^{-1} d\mathbf{P}_t(f), (B^*)^{-1} dg \rangle_B \, d\mu \, dt$$

Twisted levels

Twisted deformed parallel translation above X^x :

$$W_t^B = B(X_t)W_tB^{-1}(x).$$

Twisted generator :

$$L^{W,B} = (B^*)^{-1}L^WB^*$$

Intertwinings?

$(B^*)^{-1}dLf = L^{W,B}(B^*)^{-1}df.$

$L^{W,B}$ essentially self-adjoint : L^2-semi-group $\mathbf{Q}^B$.
With boundedness condition (on M_B but not only) : $\mathcal{C}^0$-semi-group but no intertwining proved.

Decomposition of the twisted generator

Recall :$L^W = L^{/\!/} - \mathcal{M}$.

- $L^{/\!/}$ symmetric non-positive
- $\mathcal{M}$ potential

$$L^{W,B} = \underbrace{L^{/\!/} + 2(B^*)^{-1}\nabla B^*.\nabla}_{L_B^{/\!/}} + \underbrace{(B^*)^{-1}L^{/\!/}(B^*) - (B^*)^{-1}\mathcal{M}B^*}_{-M_B}.$$

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In favor!

- M_B is the drift of W_t^B
- "differential operator" + "potential"
- heuristic

Objections ?

No linked known process. No real comprehension of where it comes from.

Proposition

For all 1-forms α, β, we have :

$$\int_M \left\langle (-L_B^{/\!/})\alpha, \beta \right\rangle_B d\mu = \int_M \langle \nabla\alpha, \nabla\beta\rangle_B \, d\mu - \int_M \langle B^*\nabla\alpha, \mathcal{B}(B^*\beta)\rangle \, d\mu$$

where $\mathcal{B} = \left((\nabla B^*)(B^*)^{-1}\right)^t - (\nabla B^*)(B^*)^{-1}$.

Corollary

If $\mathcal{B} = 0$, then $-L_B^{/\!/}$ is symmetric non-negative.

Example

$B(x) = b(x)\,\mathrm{id}_{T_xM}$. Then $M_B = (B^*)^{-1}\mathcal{M}B^* - b^{-1}L(b)\,\mathrm{id}$.

Intertwining

$\mathcal{B} = 0$: $L_B^{/\!/}$ symmetric, non-positive and M_B symmetric.

$$\rho_B = \inf_{x \in M} \left\{ \text{smallest eigenvalue of } B^* M_B (B^*)^{-1} \right\}.$$

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Theorem (H., 2019)

If $\mathcal{B} = 0$ and $-\infty < \rho_B$, then for all $f \in \mathcal{C}_c^\infty(M)$,

$$((B^*)^{-1} \mathbf{P}_t(f) = \mathbf{Q}_t^B \left((B^*)^{-1} df \right) .$$

(Sketch of the proof)

$$\begin{cases} \partial_t F = L^{W,B} F \\ F(\cdot, 0) = G \in L^2(B, \mu) \end{cases}$$

$\longrightarrow$ Uniqueness of strong solutions

Poincaré inequality

Theorem (Generalized Brascamp-Lieb inequality - H., 2019)

Assume that $\mathcal{B} = 0$ and that $\rho_B \geq 0$, then for every $f \in \mathcal{C}_c^\infty(M)$, we have :

$$\mathrm{Var}_\mu(f) \leq \int_M \langle df, \left((B^* M_B (B^*)^{-1}\right)^{-1} df\rangle \, d\mu.$$

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Corollary (Poincaré inequality)

Assuming that $\mathcal{B} = 0$ and that ρ_B is positive, for all $f \in \mathcal{C}_c^\infty(M)$, we have

$$\mathrm{Var}_\mu(f) \leq \frac{1}{\rho_B} \int_M |df|^2 \, d\mu,$$

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$$= -\int_M \left| B^*\nabla\alpha - \frac{1}{2}\mathcal{B}B^*\alpha \right|^2 d\mu + \int_M \langle B^*\alpha, N_B B^*\alpha\rangle \, d\mu$$

$$N_B(x) = \frac{1}{4}\mathcal{B}^t(x)\cdot\mathcal{B}(x) \in \mathrm{End}(T_x^* M).$$

$$\tilde{\rho}_B = \inf_{x\in M}\left\{\text{smallest eigenvalue of } \left(B^* M_B (B^*)^{-1})\right)^s - N_B\right\}.$$

Theorem (H., 2019)

Assume that $\left(B^* M_B (B^*)^{-1})\right)^s - (1+\varepsilon) N_B$ *is bounded from below for some* $\varepsilon > 0$. *Then the semi-groups* $\mathbf{P}$ *and* $\mathbf{Q}^B$ *are intertwined by* $(B^*)^{-1} d$, *i.e for every* $f \in \mathcal{C}_c^\infty(M)$ *and* $t \geq 0$ *we have :*

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$$\mathrm{Var}_\mu(f) \leq \frac{1}{\tilde{\rho}_B} \int_M |df|^2 \, d\mu.$$

In $\mathbb{R}^2$, $\sigma^2(x) = 1 + |x|^2$. $\beta > 1$

Measure :

$$d\mu_\beta = Z(\sigma^2)^{-\beta} dx$$

Generator :

$$L_\beta = \sigma^2(x)\Delta_E - 2(\beta - 1)x.\nabla_E$$

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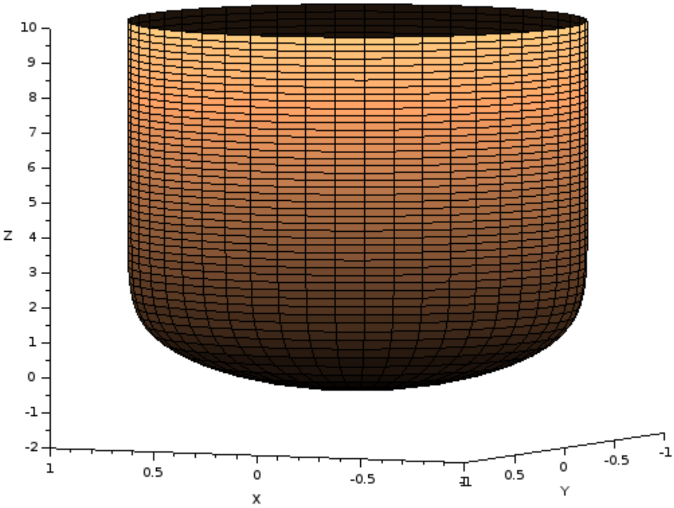
Appropriate polar coordinates :

$$(x_1, x_2) = (\sinh(r)\cos(\theta), \sinh(r)\sin(\theta)), \quad ds^2 = dr^2 + th(r)^2 d\theta^2$$

$$d\mu_\beta = Z \cosh^{-2(\beta-1)} d\,\mathrm{vol}.$$

Potential :

$$V(r) = 2(\beta - 1)\ln(\cosh(r)).$$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Bakry-Émery?

$$\mathcal{M} = \frac{2\beta}{\sigma^2}\,\mathrm{id}\,.$$

Loss of strict convexity at ∞.

Twist :

$$B = e^{\varepsilon V}\,\mathrm{id}\,.$$

$$\rho_B(r) = 2\beta - 4\varepsilon(\beta - 1) + \left[4\varepsilon(1-\varepsilon)(\beta-1)^2 - (2\beta - 4\varepsilon(\beta-1))\right]\tanh^2(r).$$

Corollary

The spectral gap of the operator L_β is bounded from below by :

$$\lambda_1(L_\beta) \geq \begin{cases} (\beta-1)^2 & \text{if} \quad 1 < \beta \leq 1+\sqrt{2} \\ 2\sqrt{(\beta-1)^2-1} & \text{if} \quad 1+\sqrt{2} \leq \beta \end{cases} .$$

Bobkov-Ledoux (2009) $\lambda_1(L_\beta) \geq \frac{2(\beta-1)}{\left(\sqrt{1+\frac{2}{\beta-1}}+\sqrt{\frac{2}{\beta-1}}\right)^2}$, $\beta \geq 2$.

Nguyen (2013) $\lambda_1(L_\beta) = 2(\beta-1)$, $\beta \geq 3$.

Bonnefont-Joulin-Ma (2016)

$$\begin{array}{rll} \lambda_1(L_\beta) = & (\beta-1)^2 & \text{if} \quad 1 < \beta \leq \frac{3+\sqrt{5}}{2} \\ \beta \leq \lambda_1(L_\beta) \leq & 2(\beta-1) & \text{if} \quad \frac{3+\sqrt{5}}{2} \leq \beta \end{array} .$$

1 Twisted intertwining and Poincaré inequality

2 **Brenier-Schrödinger problem**

- A fluid evolution problem
- Kinetic properties
- Existence of solutions

Euler equations - Newton principle :

$$\begin{cases} \partial_t v + \nabla_v v + \nabla p = 0, & (t,x) \in [0,1] \times M \\ \mathrm{div}(v) = 0, & (t,x) \in [0,1] \times M \\ \langle v, \nu \rangle = 0 & \\ v(0,\cdot) = v_0, & x \in M \end{cases}$$

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$$\Downarrow$$

Arnold minimisation problem - least action principle ('66) :

$$\int_{[0,1]\times M} |\partial_t q_t(x)|^2 \, dtdx \to \min; [q_t \in G_{\mathrm{vol}}, \ \forall 0 \le t \le 1], q_0 = \mathrm{id}, q_1 = h,$$

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$$\Downarrow$$

Brenier's relaxation ('89):

$$\mathbb{E}_Q \left[\int_0^1 |\dot{X}_t|^2 \, dt \right] \to \min; Q \in \mathcal{P}(\Omega), \left[Q_t = \mathrm{vol}, \forall 0 \le t \le 1 \right], Q_{0,1} = \pi,$$

Navier-Stokes Equations :

$$\begin{cases} \partial_t v + \nabla_v v - a\Box v + \nabla p = 0, & (t,x) \in [0,1] \times M \\ \mathrm{div}(v) = 0, & (t,x) \in [0,1] \times M \\ \langle v, \nu \rangle = 0, & (t,x) \in [0,1] \times \partial M \\ v(0,\cdot) = v_0, & x \in M \end{cases}$$

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Notion of velocity? Stochastic velocities :

$$\overset{\rightharpoonup}{v}^P_t = \lim_{h \to 0^+} \frac{1}{h} \mathbb{E}_P \left[\overrightarrow{X_t X_{t+h\wedge\tau_t}} | X_{[0,t]} \right] \quad \text{(forward)}$$

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Link : stochastic kinetic energy $\rightleftarrows$ entropy.

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framework

- M compact manifold with boundary
- R law of the reflected Brownian motion

Definition (Brenier-Schrödinger problem)

$$H(Q|R) \to \min, \; Q \in \mathcal{P}(\Omega), \; [Q_t = \mu_t, \; \forall t \in \mathcal{T}], \; Q_{01} = \pi. \tag{BS}$$

Questions

- Link between solutions of both problems.
- Existence of solutions.

Definition (Regular solution)

$$P = \exp\left(\eta(X_0, X_1) + \sum_{s \in \mathcal{S}} \theta_s(X_s) + \int_{\mathcal{T}} p_t(X_t)\,dt\right) R,$$

such that well-defined quantities and sufficient regularity.

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Previous results : (Arnaudon, Cruzeiro, Léonard, Zambrini, 2020) for $M = \mathbb{R}^n$ or $M = \mathbb{T}^n$.

Contributions : Manifold framework. Boundary behaviour

Theorem (Garcia Zelada, H., 2020)

For P_0 almost all $y \in M$, the backward stochastic velocity $\overleftarrow{v}^{y}$ satisfies :

$$\begin{cases} \left(\partial_t + \nabla_{\overleftarrow{v}^{y}}\right) \overleftarrow{v}^{y} = \frac{a}{2}\Box \overleftarrow{v}^{y} - \mathbb{1}_{\mathcal{T}}(t)\nabla ap, & 0 \leq t < 1, t \notin \mathcal{S}, z \in M, \\ \overleftarrow{v}^{y}_{t} - \overleftarrow{v}^{y}_{t^-} = \theta_t(.), & t \in S, z \in M, \\ \langle \overleftarrow{v}^{y}, \nu(z) \rangle = 0, & z \in \partial M, \\ \overleftarrow{v}^{y}_{0} = -\nabla \eta(., y), & z \in M. \end{cases} \tag{1}$$

Furthermore, there exist a scalar potential φ^y satisfying a second order Hamilton-Jacobi equation, such that

$$\overleftarrow{v}^{P}_{t}(X) = -a\nabla \varphi_t^{X_1}(X_t),\ P\text{-a.s.} \tag{2}$$

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- Impermeability condition satisfied.
- Incompressibility is not reached.

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- Impermeability condition satisfied.
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$$\overset{\hookrightarrow}{v}_t(z) = \mathbb{E}_P\left[\overset{X_0\rightharpoonup}{v}_t \,|X_t = z\right] \quad \text{and} \quad \overset{\hookleftarrow}{v}_t(z) = \mathbb{E}_P\left[\overset{\leftharpoonup X_1}{v}_t \,|X_t = z\right].$$

$$v_{cu} = \frac{1}{2}\overset{\hookrightarrow}{v}_t + \frac{1}{2}\overset{\hookleftarrow}{v}_t .$$

Theorem (Garcia Zelada, H., 2020)

Assuming that $\mathcal{T} = [0,1]$, the current velocity v_{cu}^P satisfies the continuity equation

$$\partial_t \mu_t + \mathrm{div}(\mu_t v_{cu}) = 0.$$

But does not satisfy Navier-Stokes equation.

$$H(Q|R) \to \min, \; Q \in \mathcal{P}(\Omega), \; [Q_t = \mathrm{vol}, \forall t \in [0,1]], \; Q_{01} = \pi. \qquad \text{(BS)}$$

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Previous results : (Arnaudon, Cruzeiro, Léonard, Zambrini, 2020) $M = \mathbb{T}^n$ with the necessary and sufficient condition $H(\pi|R_{0,1}) < \infty$.

Symmetric spaces

(M, g) compact Riemaniann manifold. G group of isometries. $G \curvearrowright M$ transitive.

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Examples : $\mathbb{S}^n$, $\mathbb{T}^n$.

Theorem (Garcia Zelada, H., 2020)

The Brenier-Schrödinger problem admits a unique solution if and only if $H(\pi|R_{01}) < \infty$.

Quotient spaces

$N = M/G$ with M Riemanian manifold and G reflection group.
$q : M \to N$ quotient transport :

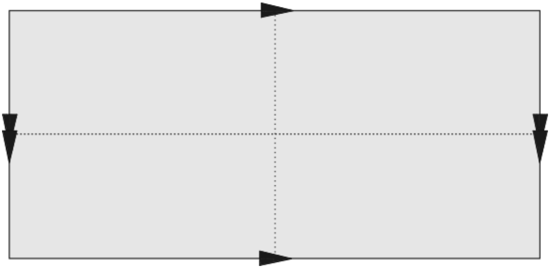
- volume measure
- Brownian motion to reflected Brownian motion
- Path measures satisfying marginals and entropy conditions

Theorem (Garcia Zelada, H., 2020)

Let π be a probability measure on $M \times M$ with both marginals equal to vol_M and such that $H(\pi|R_{01}) < \infty$. If $BS_{M,\pi}$ admits a solution, then $BS_{N,(q\times q)_*\pi}$ admits a solution. In particular, if $H_{M,\pi}$ admits a solution for every such π, then $H_{N,\tilde{\pi}}$ admits a solution for every probability measure $\tilde{\pi}$ on $N \times N$ with both marginals equal to vol_N and such that $H(\tilde{\pi}|\tilde{R}_{01}) < \infty$.

Example - Hyper-rectangle

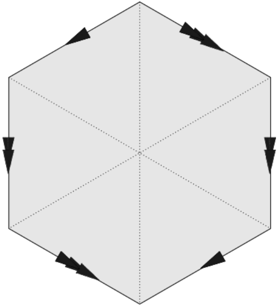
An n-dimensional rectangular box can be seen as quotient of a torus $\mathbb{T}^n$.



$\longrightarrow$ Existence and uniqueness of solution if and only if finite entropy.

Example - Equilateral triangle

An equilateral triangle can be seen as quotient space of the torus $\mathbb{T}^2$.



$\longrightarrow$ Existence and uniqueness of solution if and only if finite entropy.

Gaussian problem

$$H(P|R) \to \min; [P_t = \mathcal{N}(0, 1/4\,\mathrm{id}), \forall 0 \le t \le 1], P_{01} = \pi \qquad (BS_\gamma)$$

Theorem (Garcia Zelada, H., 2020)

The Brenier-Schrödinger problem (BS_γ) admits a unique solution if and only if $H(\pi|R_{01}) < \infty$.

Merci de votre attention.